MAT802 Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Prof. Dr. Stefan Sauter
Institut für Mathematik
Universität Zürich

Sheet 12

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Exercise 1 (2 Points)

Show that it holds

$$\|v\|^2_{H^s(\Omega)} \leq \|v\|_{H^{s-1}(\Omega)} \|v\|_{H^{s+1}(\Omega)} \quad \forall v \in H^{s+1}(\Omega) \cap H^1_0(\Omega)$$

for $s = 0$ and $s = 1$.

Dual space The dual space $H^{-1}(\Omega)$ is given by $H^{-1}(\Omega) = \left(H^1_0(\Omega)\right)'$; if $v \in L^2(\Omega)$ $\left(\hookrightarrow H^{-1}(\Omega)\right)$, it holds $\|v\|_{H^{-1}(\Omega)} = \sup_{w \in H^1_0(\Omega)} \frac{(v,w)_0}{\|w\|_{H^1}}$.

Exercise 2 (2 Points)

Let $S := \mathring{S}^k_{\mathcal{G}}$ and let $a(\cdot,\cdot) : S \to \mathbb{R}$ be a symmetric and coercive bilinear form. The scalar product $(\cdot,\cdot)_S$ and the discrete norm $|||\cdot|||_{s,S}$ are the same as in the lecture notes.

Define a mapping $B : S \to S$ via the following condition

$$\textit{For } u \in S, w := Bu \textit{ solves } (w,v)_S + a(w,v) = a(u,v) \quad \forall v \in S\,.$$

Show that

$$|||Bu|||_{s,S} \leq |||u|||_{s,S} \quad \forall s \in \mathbb{R}\,.$$

Exercise 3 (2 Points)

Let A be a symmetric and positive definite matrix, and let the decomposition be $A = M - N$ where N is also symmetric and positive definite. The linear system $Ax = r$ is solved by an iterative method $x^{(i+1)} = x^{(i)} - M^{-1}(Ax^{(i)} - r)$.

Show the convergence of the iteration by proving that the eigenvalues of the iteration matrix are real and lie in the interval $(0, 1)$.

Exercise 4 (2 Points)

With the same settings as in Theorem 10.7 in the manuscript, and given

$$\delta_0 = 0 \leq \frac{c}{c+2\nu} \text{ and } \delta_m \leq \max_{0\leq\rho\leq 1} \left\{\rho^{2\nu}[\delta_{m-1} + (1-\delta_{m-1}) \min\{1, c(1-\rho)\}]\right\}\,,$$

show by induction

$$\delta_m \leq \frac{c}{c+2\nu}\,.$$