MAT802 Numerical Methods for Elliptic and Parabolic Partial Differential Equations

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Sheet 11

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Exercise 1 (4 Points)

Consider the reaction-diffusion equation $-\nabla \cdot (\boldsymbol{A}\nabla u) + c\,u = f$ with homogeneous Dirichlet boundary condition. The coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ is symmetric and positive-definite; the coefficient c is a non-negative constant.

The finite element discretisation of the above reaction-diffusion problem yields a system matrix A. Assume the space $S := S_0^{k,0}$, the scalar product $(\cdot,\cdot)_S$ as in the lecture notes and the bilinear form $a(u,v)$ resulting from the above reaction-diffusion problem. Prove that the condition number of the system matrix A is

$$\kappa_{\|\cdot\|_S}(A) = \frac{\Lambda_{\max}}{\Lambda_{\min}}\,,$$

where $\Lambda_{\max} = \sup\limits_{v\in S\setminus\{0\}} \dfrac{a(v,v)}{(v,v)_S}$ and $\Lambda_{\min} = \inf\limits_{v\in S\setminus\{0\}} \dfrac{a(v,v)}{(v,v)_S}$.

Exercise 2 (6 Points)

The following one-dimensional problem

$$-u''(x) = f(x) \text{ for } x \in (0,1) \text{ and } u(0) = u(1) = 0$$

is discretised by using $S_{\mathcal{G}}^{1,0}$ on a uniform mesh $\mathcal{G}$ with mesh size $h = \frac{1}{N+1}$, which results in a linear system $Au_h = r$. The eigenpairs $(\lambda_\mu, v_\mu) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$ of the system matrix A are characterised by

$$Av_\mu = \lambda_\mu v_\mu \text{ for } \mu = 1, 2, \cdots, N\,.$$

The application of the Jacobi iterative method to the linear system $Au_h = r$ gives

$$u_h^{(i+1)} = u_h^{(i)} - \omega D^{-1}\left(Au_h^{(i)} - r\right) \text{ where } D = diag(diag(A))\,.$$

Denote the error after the i-th iteration by $e^{(i)} := u_h - u_h^{(i)}$, where u_h is the exact solution of the linear system. The error propagation is then expressed by

$$e^{(i+1)} = Ke^{(i)}\,.$$

a) Find the eigenvalues λ_μ and eigenvectors v_μ of the system matrix A.

b) Compute the iteration matrix K in the error propagation.

c) Compute the error $e^{(\nu)} = \sum_{\mu=1}^N c_\mu^{(\nu)} v_\mu$ after ν iterations for the given initial error $e^{(0)} = \sum_{\mu=1}^N c_\mu^{(0)} v_\mu$ with $c_\mu^{(0)} \in \mathbb{R}$.

d) Choose the damping coefficient $\omega = \frac{h}{3}$ in the Jacobi iteration.

   i) In which range of μ does it hold $\left|c_\mu^{(i+1)}\right| \le \frac{1}{2}\left|c_\mu^{(i)}\right|$?

   ii) For which $\mu \in \{1, 2, \cdots, N\}$ is the reduction factor α_μ in $\left|c_\mu^{(i+1)}\right| \le \alpha_\mu \left|c_\mu^{(i)}\right|$ smallest and largest respectively?

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e) Assume that the initial error is given by

$$e^{(0)} = (e_j)_{j=1}^{N} \text{ where } e_j = \begin{cases} 1 & \text{if } j \text{ is odd}\,, \\ 0 & \text{if } j \text{ is even}\,. \end{cases}$$

Plot the error $e^{(i)}$ for $i = 0, 1, \cdots, 5$.