[MAT121] Analysis I
Homework 11

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Deadline: Friday, 08.12.2017, 12:00
Remember to write your name and your assistant's name

(1)[15p] Compute, compute, compute

Find for each of the listed functions all points $x \in \mathbb{R}$ such that the function is differentiable and determine its derivative in these points.

1. $$f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} x^2, & x \in \mathbb{Q}; \\ 0, & x \notin \mathbb{Q}. \end{cases}$$
   One may use without proof that for every point $x \in \mathbb{R}$ exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \setminus \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = x$.
2. $$g : \mathbb{R} \to \mathbb{R},\ g(x) = |x^2 - 1|$$
3. $$h : \mathbb{R}\setminus\left(\frac{\pi}{2} + \pi\mathbb{Z}\right) \to \mathbb{R},\ h(x) = \tan(x) \cdot \ln(x^2 + 1)$$
4. $$\cosh : \mathbb{R} \to \mathbb{R},\ \cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$
5. $$j : \mathbb{R} \to \mathbb{R},\ j(x) = \sqrt[3]{x}$$

(2)[15p] Uniform limit of differentiable function is not differentiable

Let $n \in \mathbb{N}_{>0}$ and set

$$f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \begin{cases} -x, & x \le -\frac{1}{n}; \\ \frac{n}{2}x^2 + \frac{1}{2n}, & x \in (-\frac{1}{n}, \frac{1}{n}); \\ x, & x \ge \frac{1}{n}. \end{cases}$$

1. Prove f_n is differentiable.
2. Prove $(f_n)_{\mathbb{N}_{>0}}$ converges uniformly.
3. Determine all the points in which the limiting function in 2. is differentiable and compute its derivative in these points.

(3)[15p] <u>Truth or dare</u>

Decide for each statement, whether it is true or false (give proof or a counterexample)

1. If f is differentiable in x_0, then $\exists \varepsilon > 0$ such that f is differentiable on $(x_0 - \varepsilon, x_0 + \varepsilon)$.
2. If f, g are differentiable, then $\max\{f, g\}$ is differentiable.
3. If $f : \mathbb{R} \to \mathbb{R}$ is bijective and differentiable, then f^{-1} is differentiable.
4. If f is differentiable and Lipschitz, then f' is bounded.
5. If f is differentiable, then f is Lipschitz.

(4)[15p] <u>Oscillation until the bitter end</u>

Let $k \in \mathbb{R}$ and define

$$g_k : \mathbb{R} \to \mathbb{R},\ g_k(x) = \begin{cases} |x|^k \sin\left(\frac{1}{x}\right), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

1. Prove

$$\lim_{x \to 0} |x|^k = \begin{cases} 0, & k > 0; \\ 1, & k = 0; \\ \infty, & k < 0. \end{cases}$$

2. Decide for which $k \in \mathbb{R}$ the function g_k is continuous.
3. Decide for which $k \in \mathbb{R}$ the function g_k is differentiable and compute for these cases $g_k'(0)$.