Exercise Sheet 9 - Solution

Exercise 1

a) 1) $\int_{-3}^{3} x^2 dx = 2\int_{0}^{3} x^2 dx = \frac{2}{3}\Big[x^3\Big]_0^3 = \frac{2}{3}\cdot 27 = 18$

2) $$\int_{-\pi}^{\pi} \sin(x)dx = \Big[-\cos(x)\Big]_{-\pi}^{\pi} = (-\cos(\pi)) - (-\cos(-\pi))$$
$$= -\cos(\pi) + \cos(-\pi) = -(-1) + (-1) = 1 - 1 = 0$$

Alternatively, since $\sin(x)$ is a point-symmetric function one directly gets $\int_{-\pi}^{\pi} \sin(x)dx = 0$.

3) $\int_{-\pi}^{\pi} \cos(x)dx = 2\int_{0}^{\pi} \cos(x)dx = 2\Big[\sin(x)\Big]_0^{\pi} = 2(\sin(\pi) - \sin(0)) = 2(0-0) = 0$

b) Apply integration by parts: $u'(x) = \sin(x) \quad v(x) = \cos(x)$

$$\int \sin(x)\cdot\cos(x)dx = -\cos(x)\cdot\cos(x) - \int -\cos(x)\cdot(-\sin(x))dx$$
$$= -\cos^2(x) - \int \cos(x)\cdot\sin(x)dx$$

Add the integral $\int \cos(x)\cdot\sin(x)dx$ on both sides

$$2\cdot\int \sin(x)\cdot\cos(x)dx = -\cos^2(x) \quad\Rightarrow\quad \int \sin(x)\cdot\cos(x)dx = -\frac{1}{2}\cos^2(x) + C$$

Apply substitution:

$$u(x) = \sin(x) \quad\Rightarrow\quad \frac{du}{dx} = \cos(x) \quad\Rightarrow\quad du = \cos(x)dx$$
$$\int \sin(x)\cdot\cos(x)dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C$$

The results are equal since

$$\frac{1}{2}\sin^2(x) + C = \frac{1}{2}(1-\cos^2(x)) + C = -\frac{1}{2}\cos^2(x) - \frac{1}{2} + C = -\frac{1}{2}\cos^2(x) + \tilde{C}$$

Exercise 2 (5 points)

a) (1 point)

$$\int \frac{x^3+1}{\sqrt{x}}dx = \int \left(\frac{x^3}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx = \int \left(x^3 \cdot x^{-\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx$$
$$= \int \left(x^{\frac{5}{2}} + x^{-\frac{1}{2}}\right) dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{7}\sqrt{x^7} + 2\sqrt{x} + C$$
$$= \frac{2}{7}x^3\sqrt{x} + 2\sqrt{x} + C$$

b) (1 point) From the task we know that $x > 0$ must hold! Similar as in script Luchsinger page106 below and with the notation used there we need to rewrite the integrand $g(x) = \frac{1}{x\ln(x)}$ in the form $g(x) = f(u(x)) \cdot u'(x)$. We choose $f(u) = \frac{1}{u}$ and $u(x) = \ln(x)$ so that $u'(x) = \frac{1}{x}$.
So we get

$$\int \frac{dx}{x\ln(x)} = \int \frac{1}{\ln(x)} \cdot \frac{1}{x}dx = \int \frac{1}{u}du = \ln(|u|) + C = \ln(|\ln(x)|) + C$$

c) (1 point) Similarly with $g(x) = x^3(x^4+1)^{100} = \frac{1}{4} \cdot 4x^3 \cdot (x^4+1)^{100}$ we set $f(u) = u^{100}$ and $u(x) = x^4 + 1$ so that $u'(x) = 4x^3$.
So we get

$$\int x^3(x^4+1)^{100}dx = \frac{1}{4}\int (x^4+1)^{100} \cdot 4x^3 dx = \frac{1}{4}\int u^{100}du = \frac{1}{4} \cdot \frac{u^{101}}{101} + C$$
$$= \frac{1}{404} \cdot u^{101} + C = \frac{1}{404} \cdot \left(x^4+1\right)^{101} + C$$

d) (1 point) Similarly with $g(x) = \frac{x+1}{x^2+2x} = \frac{1}{2} \cdot \frac{2x+2}{x^2+2x}$ we set $f(u) = \frac{1}{2} \cdot \frac{1}{u}$ and $u(x) = x^2 + 2x$ so that $u'(x) = 2x + 2$.

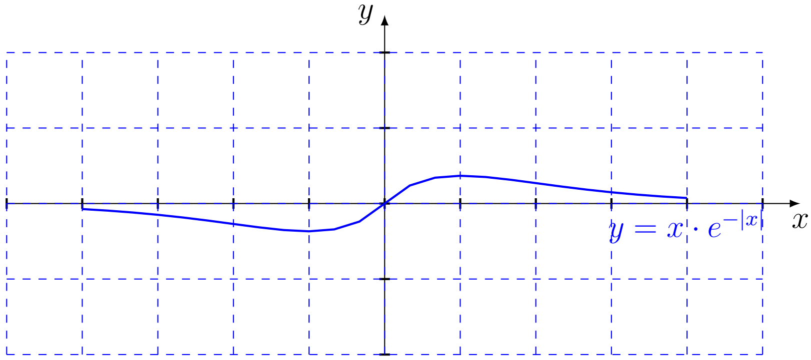
$$\int \frac{x+1}{x^2+2x}dx = \frac{1}{2}\int \frac{1}{x^2+2x} \cdot (2x+2)dx = \frac{1}{2}\int \frac{1}{u}du$$
$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+2x| + C$$

e) (1 point) Similarly with $g(x) = e^{x-e^x} = e^x \cdot e^{-e^x}$ we set $f(u) = e^{-u}$ and $u(x) = e^x$ so that $u'(x) = e^x$.

$$\int e^{x-e^x}dx = \int e^{-e^x} \cdot e^x dx = \int e^{-u}du = -e^{-u} + C = -e^{-e^x} + C$$

Exercise 3 (6 points)

a) (1 point) To integrate $e^{-|x|}$ we must distinguish between positive and negative values for x.
For $x > 0$ we have: $f(x) = x \cdot e^{-|x|} = x \cdot e^{-x}$.
For $x < 0$ we have: $f(x) = x \cdot e^{-|x|} = x \cdot e^{-(-x)} = x \cdot e^{x}$.
It holds $f(-x) = (-x) \cdot e^{-x} = -x \cdot e^{-x} = -f(x)$. The funciton is point-symmetric, see plot.



So we get $\int_{-3}^{3} x \cdot e^{-|x|}\, dx = 0$.

b) (2 points) We apply integration by parts. (1 point)

$$\int x \cdot \sin(x)\, dx = x \cdot (-\cos(x)) - \int -\cos(x)\, dx = -x \cdot \cos(x) + \int \cos(x)\, dx$$
$$= -x \cdot \cos(x) + \sin(x) + C$$ (1 point)

c) (2 points) We apply integration by parts. (1 point)

$$\begin{aligned}\int x^2 \cdot \sin(x)\, dx &= x^2 \cdot (-\cos(x)) - \int 2x \cdot (-\cos(x))\, dx \\ &= -x^2 \cdot \cos(x) + 2 \cdot \int x \cdot \cos(x)\, dx \\ &= -x^2 \cdot \cos(x) + 2 \cdot \left[x \cdot \sin(x) - \int \sin(x)\, dx \right] \\ &= -x^2 \cdot \cos(x) + 2 \cdot x \cdot \sin(x) + 2 \cdot \int -\sin(x)\, dx \\ &= -x^2 \cdot \cos(x) + 2 \cdot x \cdot \sin(x) + 2 \cdot \cos(x) + C \qquad \text{(1 point)}\end{aligned}$$

d) (1 point) The integral $\int_{-2}^{2} x^2 \sin(x)\, dx$ can be determined without integration. The funciton $f(x) = x^2 \sin(x)$ is point-symmetric, it holds $f(-x) = -f(x)$. So $\int_{-2}^{2} x^2 \sin(x)\, dx = 0$.
From the integral computed in subtaks c) it follows that the antiderivative $F(x)$ is symmetric, so $F(-x) = F(x)$. Also here we get
$\int_{-2}^{2} x^2 \sin(x)\, dx = \Big[F(x) \Big]_{-2}^{2} = F(2) - F(-2) = 0.$

Exercise 4 (2 points)

a) (1 point) With $u'(x) = x$ and $v(x) = \ln(x)$ it follows:

$$\begin{aligned}\int x \ln(x) =& \frac{1}{2} x^2 \ln(x) - \int \frac{1}{2} x^2 \frac{1}{x} dx = \frac{1}{2} x^2 \ln(x) - \frac{1}{2} \int x dx \\ =& \frac{1}{2} x^2 \ln(x) - \frac{1}{2} \cdot \frac{1}{2} x^2 + C = \frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 + C\end{aligned}$$

b) (1 point) With $u'(x) = \ln(x)$ and $v(x) = x$ it follows:

$$\int x \ln(x) = x \int \ln(x) dx - \int \left(\int \ln(x) dx \right) dx$$

Now we have to determine $\int \ln(x) dx$:

$$\begin{aligned}\int \ln(x) dx =& \int 1 \cdot \ln(x) dx = x \ln(x) - \int x \cdot \frac{1}{x} dx \\ =& x \ln(x) - \int 1 dx = x \ln(x) - x + C\end{aligned}$$

Plugging in the above integral one gets:

$$\int x\ln(x)dx = x\int \ln(x)dx - \int\left(\int \ln(x)dx\right)dx$$

$$\int x\ln(x)dx = x(x\ln(x)-x) - \int (x\ln(x)-x)\,dx$$

$$\int x\ln(x)dx = x^2\ln(x) - x^2 - \int x\ln(x)dx + \int xdx \qquad \Big| + \int x\ln(x)dx$$

$$2\cdot\int x\ln(x)dx = x^2\ln(x) - x^2 + \frac{1}{2}x^2 + \tilde{C}$$

$$2\cdot\int x\ln(x)dx = x^2\ln(x) - \frac{1}{2}x^2 + \tilde{C} \qquad \Big| \div 2$$

$$\int x\ln(x)dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

Exercise 5 (4 points)

a) (2 points) We first draw a qualitative graph that helps computing the integral.

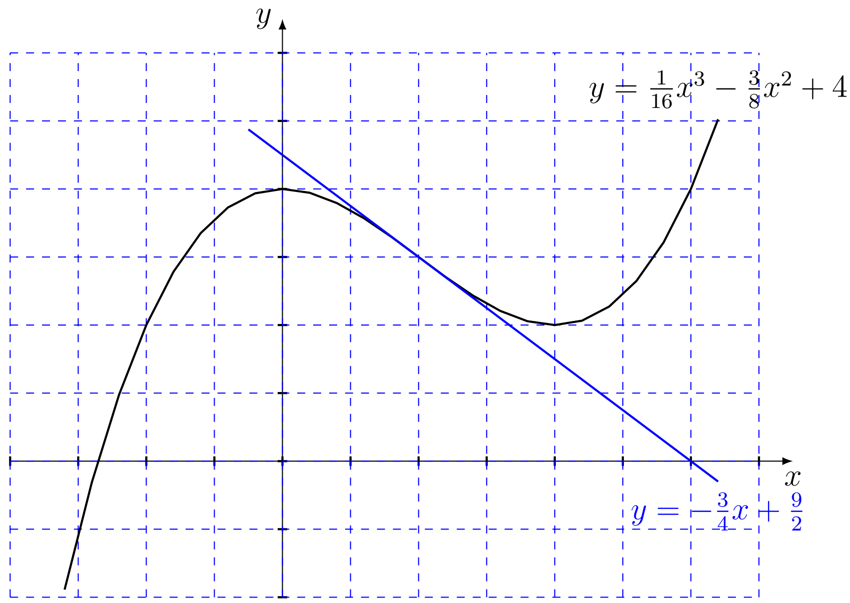
$$f(x) = \frac{1}{16}x^3 - \frac{3}{8}x^2 + 4 \qquad f'(x) = \frac{3}{16}x^2 - \frac{3}{4}x \qquad f''(x) = \frac{3}{8}x - \frac{3}{4}$$

- Roots: One root at $x_1 \approx -2,71$ is given!
- Extrema: $f'(x) = 0$. A short calculation shows: $x_2 = 0$ and $x_3 = 4$

$$f''(x_2) = -\frac{3}{4} < 0 \quad \Rightarrow \quad \text{local maximum} \quad f(x_2) = 4$$

$$f''(x_3) = \frac{3}{4} > 0 \quad \Rightarrow \quad \text{local minimum} \quad f(x_3) = 2$$

- Inflection points: $f''(x) = 0$. A short computation shows $x_4 = 2$. Further we have $f'''(x_4) \neq 0$. The slope of the tangent in the inflection point is $f'(x_4) = -\frac{3}{4}$.
  The equation of the tangent in the inflection point can be computed from $\frac{y-f(x_4)}{x-x_4} = f'(x_4)$. We find $y = -\frac{3}{4}x + \frac{9}{2}$. (1 point)



(1 point)

b) (2 points) Now we can compute the integral!

$$\begin{aligned}
F &= \int_0^2 \left(\frac{1}{16}x^3 - \frac{3}{8}x^2 + 4\right) dx + \int_2^6 \left(-\frac{3}{4}x + \frac{9}{2}\right) dx \\
&= \left[\frac{1}{64}x^4 - \frac{1}{8}x^3 + 4x\right]_0^2 + \left[-\frac{3}{8}x^2 + \frac{9}{2}x\right]_2^6 \\
&= \left[\frac{16}{64} - \frac{8}{8} + 8\right] + \left[-\frac{3\cdot 36}{8} + \frac{9\cdot 6}{2}\right] - \left[-\frac{3\cdot 4}{8} + \frac{9\cdot 2}{2}\right] \\
&= \left[\frac{1}{4} - 1 + 8\right] + \left[-\frac{3\cdot 9}{2} + 27\right] - \left[-\frac{3}{2} + 9\right] \\
&= \frac{1}{4} + 7 - 13\frac{1}{2} + 27 + \frac{3}{2} - 9 \\
&= 7\frac{1}{4} + 13\frac{1}{2} - 7\frac{1}{2} \\
&= 7\frac{1}{4} + 6 = 13\frac{1}{4}
\end{aligned}$$

Hint:
The second integral describes the area of a triangle. This can be determined by: $\frac{4\cdot 3}{2} = 6$. The calcuation is much easier than $\int_2^6 \left(-\frac{3}{4}x + \frac{9}{2}\right) dx$.