[MAT121] Analysis I
Solutions to Homework 9

Prof. Dr. Rémi Abgrall

November 24, 2017

Deadline: Friday, 24.11.2017, 12:00
Remember to write your name and your assistant's name

(1) [15p] Pointwise maxima and suprema

(a) [10p] Let f and g be two real-valued continuous functions on an interval M
Show that $x \mapsto \max(f(x), g(x))$ is a continuous function on M.

Hint: You should consider this two cases

a.1) an $x_0 \in M$, s.t. $f(x_0) = g(x_0)$

a.2) an $x_0 \in M$, s.t. $g(x_0) < f(x_0)$

(b) [5p] Find a sequence of real-valued functions f_n on $[0, 1]$ with all the following properties and show that your f_n satisfies this properties:

* $0 \leq f_n(x) \leq 1$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$,
* each function f_n is continuous,
* the function $x \mapsto \sup_{n \in \mathbb{N}} f_n(x)$ is not continuous.

Solutions:

(a) (a.1) [5p] Let $h(x) = \max(f(x), g(x))$ and use the information that f is a continuous functions, s.t. for every $\epsilon > 0$, there exists $\delta_f > 0$, s.t. for $|x - x_0| < \delta_f$ we have $|f(x) - f(x_0)| < \epsilon$. The same for g will be that for every $\epsilon > 0$, there exists $\delta_g > 0$, s.t. for $|x - x_0| < \delta_g$ we have $|g(x) - g(x_0)| < \epsilon$. (Note: ϵ is the same for both!) Suppose $x_0 \in M$, s.t. $f(x_0) = g(x_0)$. To show that h is continuous on x_0, it is straightforward to see that $|\max(f(x), g(x)) - \max(f(x_0), g(x_0))| = |h(x) - \max(h(x_0), h(x_0))| = |h(x) - h(x_0)| < \epsilon$ since $h(x_0) = g(x_0) = f(x_0)$, no matter if $h(x) = f(x)$ or $h(x) = g(x)$ as long as $|x - x_0| < \delta$.

(a.2) [5p] Let $h(x) = \max(f(x), g(x))$ and let $g(x_0) < f(x_0)$ for $x_0 \in M$. There exists a δ_0 such that $\forall x \in (x_0 - \delta_0, x_0 + \delta_0) = I_{\delta_0}$ we have $f(x) > g(x)$. We have $h(x) = f(x)$, with $x \in I_{\delta_0}$ and since f is continuous $\Rightarrow h$ is continuous.

(b) [5p] $f_n(x) = 1 - x^n, \quad x \in [0, 1]$

* $0 \leq f_n(x) \leq 1$ for all $n \in \mathbb{N}$ and for all $x \in [0,1]$, is verified since $0 \leq 1 - x^n \leq 1, \quad 0 \leq x \leq 1.$
* each function f_n is continuous, since $1 - x^n$ is a polynomial.
* the function $x \mapsto \sup_{n\in\mathbb{N}} f_n(x)$ is not continuous, as

$$\sup f_n = \sup_{n\in\mathbb{N}} 1 - x^n = \lim_{N\to\mathbb{N}} \sup_{n\leq N} 1 - x^n =$$

$$\lim_{n\to\infty} 1 - x^N = \begin{cases} 1, & \text{if } 0 \leq x < 1 \\ 0, & \text{if } x = 1. \end{cases}$$

(2) [6p] Continuous functions

Consider $f : (0,\infty) \to \mathbb{R}$ continuous and with the property that $f(x) = f(x^2)\ \forall x \in (0,\infty)$. Show that f is constant.

Hint: Show that, for all $x \in (0,\infty)$ we have $x^{1/k} \to 1\ (k \to \infty)$.

Solution:

We want to prove that $f(x) = c$, with c some constant. We now that $f(x) = f(x^2)$. The hint $x^{1/k} \to 1$ follows from combining Example 3.1.2 and Lemma 3.1.5. together with $1 \leq x^{1/k} \leq k^{1/k}$ for $x \geq 1$ and for $0 < x < 1$ noting that

$$\lim_{k\to\infty} x^{1/k} = \lim_{k\to\infty} \frac{1}{(1/x)^{1/k}} = \frac{1}{1} = 1, \quad \text{as } 1/x \geq 1.$$

Taking $x = \sqrt{y}$, we can write $f(y) = f(\sqrt{y})$ which can be further expanded as

$$f(y) = f(\sqrt{y}) = f(y^{\frac{1}{4}}) = ... = f(y^{\frac{1}{2^k}}), \quad \forall k \in \mathbb{N}.$$

Since f is continuous we have $\forall y$ that

$$\lim_{k\to\infty} f(y^{\frac{1}{2^k}}) = f(1) = c$$

Thus we have indeed that $f(x) = c\ \forall x \in (0,\infty)$.

(3) [15p] Sequences of Functions - Uniform convergence

Determine the pointwise limit of the following sequences of functions and show if the convergence is uniform or not.

(a) [5p] $f_n(x) = (1-x)^n\, x^n, \quad x \in [0,1]\,,$

(b) [5p] $g_n(x) = \begin{cases} \dfrac{1}{n}, & 0 < x < \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \leq 1 \end{cases}$.

(c) [5p] $h_n(x) = \begin{cases} -1, & -1 \leq x \leq -\frac{1}{n} \\ nx, & -\frac{1}{n} < x < \frac{1}{n} \\ 1, & \frac{1}{n} \leq x \leq 1 \end{cases}$.

Solution:

(a) [5p] We first find the pointwise limit of the sequence (f_n) which is

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} ((1-x)x)^n = 0.$$

The sequence f_n converges pointwise to 0 in $x \in [0,1]$.
For the uniform convergence

$$\lim_{n\to\infty} ||f_n - f||_\infty = \lim_{n\to\infty} ||(1-x)x||^n \leq \lim_{n\to\infty} \big(\frac{1}{4}\big)^n = 0.$$

This means that f_n is also uniformly continuous on $x \in [0,1]$.

(b) [5p] Let $x \in]0,1]$. Since $\dfrac{1}{n} \to 0$ for $n \to +\infty$ for $x \in]0,1]$

$$\lim_n g_n(x) = 0$$

s.t. the sequence (g_n) tends pointwise on $]0,1]$ to the function $g(x) = 0$.
As for the uniform convergence of the sequence (g_n) on $]0,1]$, we have the limit

$$\lim_n ||g_n - g||_\infty = \lim_n \big[\sup_{x\in]0,1]} g_n(x)\big] = \lim_n \frac{1}{n} = 0.$$

The sequence (g_n) converges uniformly to g on $]0,1]$.

(c) [5p] We first find the pointwise limit of the sequence (h_n). We will consider $x \in [0,1]$ for h_n as in the other intervals, it is very similar. For $x = 0$ we have that $h_n(0) = 0$ for any n. This means

$$\lim_{n\to\infty} h_n(0) = 0.$$

The sequence h_n converges pointwise to 0 in $x = 0$. Considering $x \in]0,1]$, since $\dfrac{1}{n} \to 0$ for $n \to +\infty$, there exists an $N \in \mathbb{N}$ s.t. for any $n \geq N$ we have $h_n(x) = 1$. So if $x \in]0,1]$, $\lim_{n\to\infty} h_n(x) = 1$. For symmetry $x \in [-1,0[$, $\lim_{n\to\infty} h_n(x) = -1$.
The sequence (h_n) tends pointwise on $[-1,1]$ to the function

$$h(x) = \begin{cases} -1, & \text{if } -1 \leq x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x \leq 1 \end{cases}$$

Since the functions h_n are continuous, while h is not on $[-1,1]$, we have that the sequence (h_n) contradicts the definition 6.6.1 , and thus does not converge uniformly to h on $[-1,1]$.

(4) [15p] Series of functions - Normal, absolute, uniform and pointwise convergence

Determine the normal, uniform, absolute and punctual convergence of the following series of functions.

(a) [5p] $\sum_{n=1}^{\infty} \dfrac{1}{n^2(1+n^2x^2)}, \quad x \in \mathbb{R}\,,$

(b) [5p] $\sum_{n=1}^{\infty}(-1)^n x^n, \quad x \in [-1,1]\,,$

(c) [5p] $\sum_{n=1}^{\infty}\Big(\dfrac{nx}{1+nx^2} - \dfrac{(n-1)x}{1+(n-1)x^2}\Big), \quad x \in \mathbb{R}\,,$

Remark to eq. 7.1. in the script:
Consider $(f_n)_{n\in\mathbb{N}}$ a sequence of $\mathbb{R}^m$ – valued functions on an interval M. Suppose there exists a convergent series $\sum_n m_n$ with $|f_n(x)| \le m_n$ for all $n \in \mathbb{N}$ and all $x \in M$.
We define the **norm** as $||f_n|| := \sup_{x\in M}|f_n(x)|$.

We say $\sum_n f_n(x)$ is normally convergent if $\sum_n ||f_n|| < \infty$ and this means that $\sum_n f_n(x)$ for $\forall x \in M$ converges absolutely, uniformly and pointwise.
Note: When $\sum_n f_n(x)$ converges uniformly, it converges also pointwise.
When $\sum_n f_n(x)$ converges absolutely, it converges also pointwise.

Solution:

(a) [5p] The series $\sum_{n=1}^{\infty}\dfrac{1}{n^2(1+n^2x^2)}, \quad x \in \mathbb{R}$ is a series of continuous functions on $\mathbb{R}$. For every $n \ge 1$ we set $f_n(x) = \dfrac{1}{n^2(1+n^2x^2)}$. Consider the normal convergence $\sum_{n=1}^{\infty}||f_n||_\infty$. We have

$$||f_n||_\infty = \sup_{x\in\mathbb{R}}|f_n(x)| = \sup_{x\in\mathbb{R}}\Big(\frac{1}{n^2(1+n^2x^2)}\Big) = \frac{1}{n^2}$$

This means that the series $\sum_{n=1}^{\infty}||f_n||_\infty = \sum_{n=1}^{\infty}\dfrac{1}{n^2}$ is convergent. Since it converges normally, it converges also uniformly, absolutely and pointwise.

(b) [5p] The series $\sum_{n=1}^{\infty}(-1)^n x^n, \quad x \in [-1,1]$ is a series of continuous functions on $[-1,1]$. For every $n \ge 1$ we set $f_n(x) = (-1)^n x^n = (-x)^n$. We can see, this is a geometric series. This means it converges pointwise only if $|-x| < 1$, i.e. $x \in (-1,1)$ with in this case the sum of the series to be $S(x) = \dfrac{1}{1+x}$. Further, for $x \in (-1,1)$ the series $\sum_{n=0}^{\infty}|(-1)^n x^n| = \sum_{n=0}^{\infty}|x|^n$ converges to $T(x) = \dfrac{1}{1-|x|}$. This means that the series converges absolutely in $(-1,1)$ to $T(x) = \dfrac{1}{1-|x|}$. Consider now the normal convergence $\sum_{n=1}^{\infty}||f_n||_\infty$, we have

$$||f_n||_\infty = \sup_{x\in(-1,1)}|f_n(x)| = \sup_{x\in(-1,1)}|x|^n = 1,$$

i.e. $\sum_{n=1}^{\infty}||f_n||_\infty$ diverges thus is not converging normally on $(-1,1)$. For the uniform convergence, since the series S is not limited on $(-1,1)$, this means that the series does not converge uniformly on $(-1,1)$.

(c) [5p] The series $\sum_{n=1}^{\infty}\Big(\dfrac{nx}{1+nx^2} - \frac{(n-1)x}{1+(n-1)x^2}\Big), \quad x \in \mathbb{R}$ is a series of continuous functions on $\mathbb{R}$. For every $n \ge 1$ we have $f_n = \dfrac{nx}{1+nx^2} - \frac{(n-1)x}{1+(n-1)x^2}$. We can see that it is a telescopic series.

For the pointwise convergence we can see that the partial $n-th$ sum is

$$S(x)=\sum_{k=1}^{n} f_k(x)=\sum_{k=1}^{n}\Big(\frac{kx}{1+kx^2}-\frac{(k-1)x}{1+(k-1)x^2}\Big)$$
$$=\frac{x}{1+x^2}+\frac{2x}{1+4x^2}-\frac{x}{1+x^2}+...+\frac{nx}{1+nx^2}-\frac{(n-1)x}{1+(n-1)x^2}=\frac{nx}{1+nx^2}$$

The sum of this series is thus

$$S(x)=\lim_n S_n(x)=\lim_n \frac{nx}{1+nx^2}=\begin{cases}0, & \text{if } x=0\\ \dfrac{1}{x}, & \text{else}\end{cases}$$

The series converges pointwise on $\mathbb{R}$ to the function S. Since f_n is continuous on $\mathbb{R}$, also S_n is continuous on $\mathbb{R}$, while S is not continuous on 0. This last means, that the series (S_n) does not converge uniformly and normally on $\mathbb{R}$. For the absolute convergence observe that

$$f_n(x)=\frac{nx}{1+nx^2}-\frac{(n-1)x}{1+(n-1)x^2}=\frac{x}{(1+nx^2)[1+(n-1)x^2]},$$

so that $f_n\geq 0$ if and only if $x\geq 0$. Moreover we observe that if $x\geq 0$, then the series $\sum_{n=1}^{\infty}|f_n(x)|$ converges to $S(x)$. If $x<0$, then the series $\sum_{n=1}^{\infty}|f_n(x)|=\sum_{n=1}^{\infty}|f_n(-x)|$ converges to $S(-x)=-\frac{1}{x}$. This means that the series converges absolutely on $\mathbb{R}$ to

$$T(x)=\begin{cases}0, & \text{if } x=0\\ \dfrac{1}{|x|}, & \text{else}\end{cases}$$

(5) [9p] Power series

Prove that the following power series (which sum is $\sin z$) has a convergence radius of $+\infty$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}z^{2n+1}, \quad z\in\mathbb{C}.$$

Solution:

Take $z\in\mathbb{C}$, $z\neq 0$. We study the absolute convergence of the power series in z, i.e. $\sum_{n=0}^{\infty}\frac{|z|^{2n+1}}{(2n+1)!}$. Applying the ratio criteria

$$\frac{\frac{|z|^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{|z|^{2n+1}}{(2n+1)!}}=\frac{|z|^{2n+3}}{(2n+3)!}\frac{(2n+1)!}{|z|^{2n+1}}=|z|^2\frac{1}{(2n+1)!(2n+2)(2n+3)}(2n+1)!$$
$$=\frac{|z|^2}{(2n+2)(2n+3)}\leq\frac{|z|^2}{4n^2}\to_{n\to\infty}0.$$

The series is thus absolute convergent in z. For the arbitrariness of z, the power series has convergence radius $=+\infty$, since (following a Remark after Satz 7.3.1 in the script) the convergence radius is equal to $\sup\{|z|\in\mathbb{R}_{\geq 0}\mid \sum_{n=0}^{\infty}a_nz^n \text{ converges }\}$.