[MAT121] Analysis I
Homework 9

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Deadline: Friday, 24.11.2017, 12:00
Remember to write your name and your assistant's name

(1) [15p] Pointwise maxima and suprema

(a) [10p] Let f and g be two real-valued continuous functions on an interval M. Show that $x \mapsto \max(f(x), g(x))$ is a continuous function on M.

Hint: You should consider this two cases

a.1) an $x_0 \in M$, s.t. $f(x_0) = g(x_0)$

a.2) an $x_0 \in M$, s.t. $g(x_0) < f(x_0)$

(b) [5p] Find a sequence of real-valued functions f_n on $[0, 1]$ with all the following properties and show that your f_n satisfies this properties:

* $0 \leq f_n(x) \leq 1$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$,
* each function f_n is continuous,
* the function $x \mapsto \sup_{n \in \mathbb{N}} f_n(x)$ is not continuous.

(2) [6p] Continuous functions

Consider $f : (0, \infty) \to \mathbb{R}$ continuous and with the property that $f(x) = f(x^2)$ $\forall x \in (0, \infty)$. Show that f is constant.

Hint: Show that, for all $x \in (0, \infty)$ we have $x^{1/k} \to 1$ $(k \to \infty)$.

(3) [15p] Sequences of Functions - Uniform convergence

Determine the pointwise limit of the following sequences of functions and show if the convergence is uniform or not.

(a) [5p] $f_n(x) = (1 - x)^n \, x^n, \quad x \in [0, 1]\,,$

(b) [5p] $g_n(x) = \begin{cases} \dfrac{1}{n}, & 0 < x < \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \leq 1 \end{cases} \quad ,$

(c) **[5p]** $h_n(x) = \begin{cases} -1, & -1 \leq x \leq -\frac{1}{n} \\ nx, & -\frac{1}{n} < x < \frac{1}{n} \\ 1, & \frac{1}{n} \leq x \leq 1 \end{cases}$.

(4) [15p] Series of functions - Normal, absolute, uniform and pointwise convergence

Determine the normal, uniform, absolute and punctual convergence of the following series of functions.

(a) **[5p]** $\sum_{n=1}^{\infty} \frac{1}{n^2(1+n^2x^2)}, \quad x \in \mathbb{R}\,,$

(b) **[5p]** $\sum_{n=1}^{\infty} (-1)^n x^n, \quad x \in [-1,1]\,,$

(c) **[5p]** $\sum_{n=1}^{\infty} \left(\frac{nx}{1+nx^2} - \frac{(n-1)x}{1+(n-1)x^2}\right), \quad x \in \mathbb{R}\,,$

Remark to eq. 7.1. in the script:
Consider $(f_n)_{n\in\mathbb{N}}$ a sequence of $\mathbb{R}^m$ – valued functions on an interval M. Suppose there exists a convergent series $\sum_n m_n$ with $|f_n(x)| \leq m_n$ for all $n \in \mathbb{N}$ and all $x \in M$.
We define the **norm** as $||f_n|| := \sup_{x\in M} |f_n(x)|$.

We say $\sum_n f_n(x)$ is normally convergent if $\sum_n ||f_n|| < \infty$ and this means that $\sum_n f_n(x)$ for $\forall x \in M$ converges absolutely, uniformly and pointwise.
Note: When $\sum_n f_n(x)$ converges uniformly, it converges also pointwise.
When $\sum_n f_n(x)$ converges absolutely, it converges also pointwise.

(5) [9p] Power series

Prove that the following power series (which sum is $\sin z$) has a convergence radius of $+\infty$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}, \quad z \in \mathbb{C}.$$