[MAT121] Analysis I
Homework 8

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Deadline: Friday, 17.11.2017, 12:00

Remember to write your name and your assistant's name

(1) [10p] Extreme Value Theorem

- **(a)[4p]** Give an example of a function defined on [0,1] which has no maximum or minimum on that interval.
- **(b)[6p]** If f, g are continuous functions on $[0, 1]$, let $||f||$ and $||g||$ be the maximum values of $|f|$ and $|g|$ on $[0, 1]$ respectively. Prove that $||f + g|| \leq ||f|| + ||g||$. Give an example where $||f + g|| \neq ||f|| + ||g||$.

(2) [12p] Continuous and Discontinuous Functions

- **(a)[6p]** Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ but is continuous at all other points.
- **(b)[6p]** Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ and at 0 but is continuous at all other points.

(3) [18p] $\delta - \epsilon$ Definition of Continuity

- **(a)[6p]** Suppose that f is a function that satisfies $|f(x)| \leq |x|$ for all x. Show that f is continuous at 0.
- **(b)[6p]** Suppose that g is continuous at 0 and $g(0) = 0$, and $|f(x)| \leq |g(x)|$ for all x. Show that f is continuous at 0.
- **(c)[6p]** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function which satisfies

$$\lim_{h \to 0}[f(x + h) - f(x - h)] = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

(4) [20p] Intermediate Value Property

- **(a)[6p]** Suppose that f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some x in [a,b].
- **(b)[6p]** Suppose that f has the property that for any two points $x_1 < x_2$ and any number c between $f(x_1)$ and $f(x_2)$ there is some point x in (x_1, x_2) such that $f(x) = c$. Is it true that f must be continuous? (If yes prove it, otherwise give a counterexample.)
- **(c)[8p]** Suppose that f has the property that for any two points $x_1 < x_2$ and any number c between $f(x_1)$ and $f(x_2)$ there is some point x in (x_1, x_2) such that $f(x) = c$. In addition, assume that f is monotonic. Is it true that f must be continuous? (If yes prove it, otherwise give a counterexample.)