[MAT121] Analysis I
Solution to Homework 5

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(1) [10p] Sequences I

Decide with reasoning if the following sequences converge or not. If a sequence converges, determine its limit.
Remark: Use only the definition of a converging sequence for this exercise.

(a) $a_n = \frac{2(n+7)}{n^2+1}$

(b) $b_n = \frac{5n^3-1}{6n^3+4n^2+n+2}$

(c) $c_n = (-n)^n$

(d) $d_n = (-n)^{-n}$

(e) $e_n = \frac{\sqrt{n}}{n}$

Solution:

(a) We can write

$$a_n = \frac{2(n+7)}{n^2+1} = \frac{2}{n+\frac{1}{n}} + \frac{7}{n^2+1}$$

Let $\epsilon > 0$ be given. Let $n_0 \in \mathbb{N}$ be such that for all $n \geq n_0$ it is $\frac{7}{n} < \frac{\epsilon}{2}$. Then for all $n \geq n_0$ we have

$$|\frac{2}{n+\frac{1}{n}} + \frac{7}{n^2+1}| \leq |\frac{2}{n+\frac{1}{n}}| + |\frac{7}{n^2+1}| \leq |\frac{2}{n}| + |\frac{7}{n}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

By the definition of convergence this means that $a_n \to 0$ if $n \to \infty$, i.e. a_n converges with limit 0.

(b) Again we reformulate

$$b_n = \frac{5n^3-1}{6n^3+4n^2+n+2} = \frac{5}{6+4\frac{1}{n}+\frac{1}{n^2}+2\frac{1}{n^3}} - \frac{1}{6n^3+4n^2+n+2}$$

Let $\epsilon > 0$ be given. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n} < \epsilon$ for all $n \geq n_0$. Then we get for $n \geq n_0$ that

$$|\frac{5}{6+4\frac{1}{n}+\frac{1}{n^2}+2\frac{1}{n^3}} - \frac{1}{6n^3+4n^2+n+2} - \frac{5}{6}| \leq |\frac{30-5(6+4\frac{1}{n}+\frac{1}{n^2}+2\frac{1}{n^3})}{6(6+4\frac{1}{n}+\frac{1}{n^2}+2\frac{1}{n^3})}| + |\frac{1}{n}|$$
$$\leq \frac{5(4\frac{1}{n}+\frac{1}{n^2}+2\frac{1}{n^3})}{36} + \frac{\epsilon}{2} \leq \frac{5(7\frac{1}{n})}{36} + \frac{\epsilon}{2} \leq \frac{35}{36n} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

In particular b_n converges with limit $\frac{5}{6}$.

(c) We show that for each $c \in \mathbb{R}$ there is a $n_c \in \mathbb{N}$ such that there are infinitely many $n_i > n_c$ with $|c_{n_i} - c| > 1$. This shows that no $c \in \mathbb{R}$ can be the limit of c_n and hence c_n does not converge. So let $c \in \mathbb{R}$ be given. Then there is $n_c \in \mathbb{N}$ such that $n_c > c + 1$. We get

$$(-2n)^{2n} = (2n)^{2n} > 2n > n_c > c + 1$$

for all $n > n_c$. Hence $|c_{2n} - c| > 1$ for all $n > n_c$. Hence c_n is not convergent. Also c_n is also not diverging to $\pm\infty$ since $c_{2n} \geq 0$ and $c_{2n+1} \leq 0$ for all $n \in \mathbb{N}$.

(d) Let $\epsilon > 0$ be given and $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for all $n \geq n_0$. Then

$$|(-n)^{-n}| = \frac{1}{n^n} \leq \frac{1}{n} < \epsilon$$

for all $n \geq n_0$. Hence d_n converges to 0.

(e) We reformulate $e_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. Let $\epsilon > 0$ be given. Then be $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon^2$ for all $n \geq n_0$, i.e. $\frac{1}{\sqrt{n}} < \epsilon$. Then $|e_n| = \frac{1}{\sqrt{n}} < \epsilon$ for all $n \geq n_0$ and hence e_n converges to 0.

(2) [15p] Sequences II

Decide with reasoning if the following sequences converge or not. If a sequence converges, determine its limit.

(a) $a_n = \frac{\sqrt{n}+1}{\sqrt{n}-1}$

(b) $b_n = \frac{n^n}{kn!}$ with $k \in \mathbb{N}\backslash\{0\}$

(c) $c_n = \sqrt[n]{a^n + b^n}$ with $0 < b < a$

(d) $d_n = \frac{1}{2^n}\binom{n}{k}$ for $k \in \mathbb{N}$ and $n \geq k$

(e) $e_n = \sqrt{n(n+a)} - n$ with $a \in \mathbb{R}^+$

Solution:

(a) We reformulate $a_n = \frac{\sqrt{n}+1}{\sqrt{n}-1} = \frac{1}{1-\frac{1}{\sqrt{n}}} + \frac{1}{\sqrt{n}-1}$. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{\sqrt{n}-1} < \frac{\epsilon}{2}$ for all $n \geq n_0$, i.e. $n_0 > (\frac{2}{\epsilon} + 1)^2$. Then for $n \geq n_0$

$$|a_n - 1| = |\frac{1}{1-\frac{1}{\sqrt{n}}} + \frac{1}{\sqrt{n}-1} - 1| \leq |\frac{1-(1-\frac{1}{\sqrt{n}})}{1-\frac{1}{\sqrt{n}}}| + |\frac{1}{\sqrt{n}-1}| < |\frac{1}{\sqrt{n}-1}| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence a_n converges to 1.

(b) Since $n!$ consists of a product of $n-1$ numbers smaller or equal than n (and 1) we can derive that $\frac{n^{n-1}}{n!} > 1$. More precisely $\frac{n^{n-1}}{n!} = \prod_{i=0}^{n-2} \frac{n}{n-i} > 1$. Hence $b_n = \frac{n}{k}\frac{n^{n-1}}{n!} > \frac{n}{k}$. Since k is fixed, $\frac{n}{k}$ diverges to infinity when n grows to infinity and therefore also b_n.

(c) We reformulate $c_n = \sqrt[n]{a^n + b^n} = \sqrt[n]{a^n}\sqrt[n]{1 + (\frac{b}{a})^n} = a\sqrt[n]{1 + (\frac{b}{a})^n}$. Since $0 < b < a$, we have that $\frac{b}{a} < 1$ and hence $(\frac{b}{a})^n \to 0$ and thus $1 + (\frac{b}{a})^n \to 1$, as we know from the lecture. Also we know that then $\sqrt[n]{1 + (\frac{b}{a})^n} = (1 + (\frac{b}{a})^n)^{\frac{1}{n}} \to 1^{\frac{1}{n}} = 1$. Altogether we get $c_n \to a$.

(d) We can see that

$$d_{n+1} = \frac{1}{2^{n+1}}\binom{n+1}{k} = \frac{(n+1)!}{2^{n+1}k!(n+1-k)!} = \frac{n+1}{2(n+1-k)}\frac{n!}{2^n k!(n-k)!} = \frac{n+1}{2(n+1-k)}d_n$$

Let $n_0 \in \mathbb{N}$ such that $n_0 > 2k+1$. Then $\frac{n}{n-k} \leq \frac{n_0}{n_0-k} < 2$ for all $n \geq n_0$ (the first equality follows from $\frac{n}{n-k} - \frac{n_0}{n_0-k} = \frac{-nk+n_0k}{(n-k)(n_0-k)} \leq 0$) and for such n we get

$$|d_n| = d_n \leq q^{n-n_0} d_{n_0}$$

with $q = \frac{n_0}{2(n_0-k)} < 1$. From the lecture we know that q^{n-n_0} converges to 0, and hence also $q^{n-n_0}d_{n_0}$. Furthermore $0 \leq d_n \leq q^{n-n_0}d_{n_0}$ for $n > n_0$. The right hand side converging to 0 implies that d_n converges to 0.

(e) Let $\epsilon > 0$ be given. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that $2n\epsilon > (\frac{1}{2}a - \epsilon)^2$. Then

$$2n\epsilon > (\frac{1}{2}a - \epsilon)^2 \iff n^2 + na > n^2 + na - 2n\epsilon + (\frac{1}{2}a - \epsilon)^2 \iff n^2 + na > (n + \frac{1}{2}a - \epsilon)^2$$
$$\iff \sqrt{n^2 + na} > n + \frac{1}{2}a - \epsilon.$$

Furthermore $n^2 + na + \frac{1}{4}a^2 > n^2 + na \iff n + \frac{1}{2}a > \sqrt{n^2+na}$. Altogether we get for $n \geq n_0$

$$n + \frac{1}{2}a > \sqrt{n^2+na} > n + \frac{1}{2}a - \epsilon \iff \frac{1}{2}a > \sqrt{n^2+na} - n > \frac{1}{2}a - \epsilon.$$

This yields $|e_n - \frac{1}{2}a| = |\sqrt{n^2+na} - n - \frac{1}{2}a| < \epsilon$ if $n \geq n_0$. Hence e_n converges to $\frac{1}{2}a$.

(3) [6p] Cluster points

(a)[3p] Consider the sequence $a_n = (-1)^n + (-1)^{n^2}$. Determine all cluster points of a_n and decompose the sequence into subsequences converging to cluster points. Here, decomposing means that each element of the original sequence is contained in exactly one subsequence.

(b)[3p] Decide with reasoning if every sequence in $\mathbb{R}$ can be decomposed into converging subsequences.

Solution:

(a) If n is even i.e. $n = 2k$ for some $k \in \mathbb{N}$, then is $n^2 = 4k^2$ also even and hence $a_{2k} = (-1)^{2k} + (-1)^{4k^2} = 1 + 1 = 2$. In the same way, if n is odd, i.e. $n = 2k+1$ for some $k \in \mathbb{N}$, then $n^2 = 4k^2 + 4k + 1$ and hence n^2 is also odd. We get $a_{2k+1} = (-1)^{2k+1} + (-1)^{4k^2+4k+1} = -1 - 1 = -2$. This means that the sequence a_n takes only the values -2 and 2. Hence the cluster points can only be -2 and 2, since for every $x \in \mathbb{R}$ with $x \neq -2$ and $x \neq 2$ we find $\epsilon > 0$ such that $\min\{|2-x|, |-2-x|\} > \epsilon$ and hence $|a_n - x| > \epsilon$ for all $n \in \mathbb{N}$.
If we consider the subsequence a_{2k} for $k \in \mathbb{N}$, then this sequence is constant 2 and hence converges to 2. If we consider the subsequence a_{2k+1} for $k \in \mathbb{N}$, then this sequence is constant -2 and hence converges to -2. Since every $n \in \mathbb{N}$ is either even or odd, we get a decomposition of the original sequence into two converging subsequences.

(b) Not every sequence in $\mathbb{R}$ can be decomposed into converging subsequences. Take for example the sequence $b_n = (-1)^n n$. Then b_n has no converging subsequence, since for every $b \in \mathbb{R}$ there is a $n_0 \in \mathbb{N}$ such that $|b_n| > |b+1|$ for all $n \geq n_0$, i.e. no $b \in \mathbb{R}$ can be a limit of a subsequence of b_n. Hence b_n can not be decomposed into converging subsequences.

(4) [15p] Limits

(a)[4p] Let a_n be a bounded sequence in $\mathbb{R}$ with exactly one cluster point $a \in \mathbb{R}$. Show that a_n converges with limit a.

(b)[6p] Let b_n be a convergent sequence in $\mathbb{R}$. Show that

$$\hat{b}_n = \frac{1}{n}\sum_{i=1}^{n} b_i$$

is a convergent sequence and determine its limit.

(c)[5p] Let c_n be a sequence in $\mathbb{R}$ such that

$$\hat{c}_n = \frac{1}{n}\sum_{i=1}^{n} c_i$$

converges. Decide with reasoning if c_n has to be convergent.

Solution:

(a) Let a_n be a bounded sequence in $\mathbb{R}$ with one cluster point $a \in \mathbb{R}$. Assume a_n does not converge with limit a, i.e. there exists a $\epsilon > 0$ such that there are infinitely many $n_i \in \mathbb{N}$ such that $|a_{n_i} - a| > \epsilon$.
This gives a subsequence, i.e. we get a injective map $\phi : \mathbb{N} \to \mathbb{N}$, $\phi(i) \mapsto n_i$, such that $n_i < n_j$ if $i < j$.
If we consider the sequence $\hat{a}_i := a_{n_i}$, then $\hat{a}_i$ is a bounded sequence in $\mathbb{R}$, since a_n is bounded. Hence $\hat{a}_i$ has a cluster point $\hat{a} \in \mathbb{R}$. Since $|\hat{a}_i - a| > \epsilon$, it follows that $\hat{a} \neq a$. As $\hat{a}_i$ is a subsequence of a_n, clearly $\hat{a}$ is also a cluster point of a_n.
This however contradicts the fact that a_n has only one cluster point and thus the assumption there exists a $\epsilon > 0$ such that there are infinitely many $n_i \in \mathbb{N}$ such that $|a_{n_i} - a| > \epsilon$ was wrong. This implies for all $\epsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n > n_0$, i.e. $a_n \to a$.

(b) Let $b \in \mathbb{R}$ be the limit of the convergent sequence b_n. We show that b is also the limit of $\hat{b}_n$:
Let $\epsilon > 0$ be given. Let $n_1 \in \mathbb{N}$ be such that $|b_n - b| < \frac{\epsilon}{3}$ for all $n \geq n_1$, which exists as b is the limit of b_n. Let $n_2 \in \mathbb{N}$ be such that $|\frac{1}{n}\sum_{i=1}^{n_1} b_n| < \frac{\epsilon}{3}$ for all $n \geq n_2$ and $n_3 \in \mathbb{N}$ such that $\frac{n_1 b}{n} < \frac{\epsilon}{3}$ for all $n \geq n_3$. Then for all $n \geq \max\{n_1, n_2, n_3\}$ we have

$$\begin{aligned}
|\hat{b}_n - b| &= |\frac{1}{n}((\sum_{i=1}^{n} b_i) - nb)| = |\frac{1}{n}\sum_{i=1}^{n}(b_i - b)| = |\frac{1}{n}\sum_{i=1}^{n_1}(b_i - b) + \frac{1}{n}\sum_{i=n_1+1}^{n}(b_i - b)| \\
&\leq |\frac{1}{n}\sum_{i=1}^{n_1} b_i| + |\frac{1}{n}\sum_{i=1}^{n_1} b| + \frac{1}{n}\sum_{i=n_1+1}^{n} |b_i - b| < |\frac{1}{n_2}\sum_{i=1}^{n_1} b_i| + |\frac{1}{n_3} n_1 b| + \frac{1}{n-n_1}\sum_{i=n_1+1}^{n}\frac{\epsilon}{3} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{1}{n-n_1}(n-n_1)\frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Therefore $\hat{b}_n$ converges with limit b.

(c) Consider the sequence $c_n = (-1)^n$. We know from the lecture that c_n is not converging, as it has the two cluster points -1 and 1. We claim that for $n \in \mathbb{N}$ even, i.e. $n = 2k$ for some $k \in \mathbb{N}$, $\hat{c}_n = \hat{c}_{2k} = 0$ and for $n \in \mathbb{N}$ odd, i.e. $n = 2k+1$ for some $k \in \mathbb{N}$, $\hat{c}_n = \hat{c}_{2k+1} = \frac{-1}{2k+1}$.
We show the claim by induction over $k \in \mathbb{N}$: We can just evaluate to see that $c_1 = -1$ and $c_2 = 0$.

Assume now that $\hat{c}_{2j} = 0$ and $\hat{c}_{2j+1} = \frac{-1}{2j+1}$ for all $j \leq k$. Then $\hat{c}_{2k+2} = \frac{2k}{2k+2}\hat{c}_{2k} - \frac{1}{2k+2} + \frac{1}{2k+2} = 0$ and $\hat{c}_{2k+1} = \frac{2k}{2k+1}\hat{c}_{2k} - \frac{1}{2k+1} = -\frac{1}{2k+1}$ and therefore the claim follows.
We know from the lecture that the sequence $-\frac{1}{2k+1}$ for $k \in \mathbb{N}$ converges to 0. Then it follows that $\hat{c}_n$ converges to 0. In particular we found a sequence c_n which is not converging, but $\hat{c}_n$ is.

(5) [14p] Cauchy sequences

(a)[4p] Let $x \in \mathbb{R}$ and let a_n be the sequence in $\mathbb{R}$ defined by $a_n = \frac{p_n}{n}$ with $p_n \in \mathbb{Z}$ such that $\frac{p_n}{n} \leq x < \frac{p_n+1}{n}$. Show that a_n is a Cauchy sequence. What is the limit?

(b)[2p] Prove by induction that

$$\sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q}$$

and derive that for q with $0 < q < 1$ that $|\sum_{i=0}^{n} q^i|$ is bounded independently of n.

(c)[4p] Let $0 < \theta < 1$ and c_n be a sequence in $\mathbb{R}$ such that $|c_{n+2} - c_{n+1}| < \theta|c_{n+1} - c_n|$ for all $n \in \mathbb{N}$. Show that c_n is a Cauchy sequence.

(d)[4p] Let k_n be a sequence of elements in $\mathbb{N}\backslash\{0\}$. Assign to k_n the sequence

$$d_n := (\frac{1}{2})^{k_1} + (\frac{1}{2})^{k_1+k_2} + \ldots + (\frac{1}{2})^{k_1+k_2+\ldots+k_n}.$$

Show that d_n is a Cauchy sequence.

Solution:

(a) By the definition of a_n we know that $|a_n - x| < \frac{1}{n}$, since $|x - a_n| = x - \frac{p_n}{n} < \frac{p_n+1}{n} - \frac{p_n}{n} = \frac{1}{n}$. Since for every $\epsilon > 0$, we find $n_0 \in \mathbb{N}$ such that $|x - a_n| < \frac{1}{n} < \epsilon$ for $n > n_0$, it follows that a_n converges to x and hence by a theorem of the lecture a_n is a Cauchy sequence.
Alternatively: Given $\epsilon > 0$. We can estimate that

$$|a_n - a_m| \leq |a_n - x| + |x - a_m| < \frac{1}{n} + \frac{1}{m} < \epsilon$$

If $n, m > n_0$ where n_0 such that $\frac{1}{n_0} < \frac{\epsilon}{2}$.

(b) We check $\sum_{i=0}^{0} q^i = q^0 = 1 = \frac{1-q}{1-q}$. Assume now that $\sum_{i=0}^{n-1} q^i = \frac{1-q^n}{1-q}$. Then

$$\sum_{i=0}^{n} q^i = \sum_{i=0}^{n-1} q^i + q^n = \frac{1-q^n}{1-q} + q^n = \frac{1-q^n+q^n(1-q)}{1-q} = \frac{1-q^{n+1}}{1-q},$$

proving the statement. Since we have assumed $0 < q < 1$, it follows that $0 \leq 1 - q^{n+1} \leq 1$, $0 \leq 1 - q \leq 1$ and hence $|\sum_{i=0}^{n} q^i| = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$.

(c) We first remark that we can show inductively that $|c_{n+1} - c_n| < \theta^n|c_1 - c_0|$. We set $C := |c_1 - c_0|$. Let $m, n \in \mathbb{N}$ with $m > n$. Then

$$|c_m - c_n| = |c_m - c_{m-1} + c_{m-1} - c_n| \leq |c_m - c_{m-1}| + |c_{m-1} - c_n|$$

Inductively it follows that

$$\begin{aligned}|c_m - c_n| &\leq |c_m - c_{m-1}| + |c_{m-1} - c_{m-2}| + \ldots + |c_{n+1} - c_n| \\ &\leq \theta^{m-1}C + \theta^{m-2}C + \ldots + \theta^n C = \theta^n C \sum_{i=0}^{m-n-1} \theta^i\end{aligned}$$

We can use part **(b)** of the exercise to derive that $\sum_{i=0}^{m-n-1} \theta^i \leq \frac{1}{1-\theta}$.
Let $\epsilon > 0$ be given. As $0 < \theta < 1$ we know from the lecture that $\theta^n \to 0$, therefore we find $n_0 \in \mathbb{N}$ such that $\theta^n < \epsilon \frac{1-\theta}{C}$ for all $n \geq n_0$. This gives that for all $n, m \geq n_0$ with $m > n$ we get

$$|c_m - c_n| \leq \theta^n \frac{C}{1-\theta} < \epsilon \frac{1-\theta}{C} \frac{C}{1-\theta} = \epsilon$$

and therefore c_n is a Cauchy sequence.

(d) By the definition of the sequence, we have for $m > n$ that

$$|d_m - d_n| = \sum_{i=n+1}^{m} (\frac{1}{2})^{k_1+k_2+\ldots+k_i} \leq \sum_{i=j_0}^{j_1} (\frac{1}{2})^i = (\frac{1}{2})^{j_0} \sum_{i=0}^{j_1-j_0} (\frac{1}{2})^i$$

with $j_0 := k_1 + k_2 + \ldots + k_{n+1}$ and $j_1 := k_1 + k_2 + \ldots + k_m$. We use again part **(b)** of the exercise to derive that $\sum_{i=0}^{j_1-j_0} (\frac{1}{2})^i \leq \frac{1}{1-\frac{1}{2}} = 2$. Let $\epsilon > 0$ be given. Let $n_0 \in \mathbb{N}$ such that $(\frac{1}{2})^{k_1+k_2+\ldots+k_{n_0+1}} < \frac{\epsilon}{2}$. It follows that then also $(\frac{1}{2})^{k_1+k_2+\ldots+k_n} < \frac{\epsilon}{2}$ for $n \geq n_0$. Let $n, m \geq n_0$ with $m > n$. Then

$$|d_m - d_n| \leq 2(\frac{1}{2})^{k_1+k_2+\ldots+k_{n+1}} < \frac{\epsilon}{2} 2 = \epsilon$$

and hence d_n is a Cauchy sequence.