[MAT121] Analysis I
Solutions to homework 6

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(1) **[16p]** Convergence of series

Show explicitly whether the following series are convergent or divergent:

(1.1) **[4p]** $\sum_{n=1}^{\infty}\left(\frac{4n^2+6n+1}{3n^2+2}-1\right)^n$,

(1.2) **[4p]** $\sum_{n=1}^{\infty}(-1)^n\,\frac{2^n\,n!}{n^n}$,

(1.3) **[4p]** $\sum_{n=1}^{\infty}\frac{(3n)!+4^{n+1}}{(3n+1)!}$,

(1.4) **[4p]** $\sum_{n=1}^{\infty}\frac{\sqrt{n^4-1}}{n^3}$,

Solution:

(1.1) **[4p]** We have that,

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=\lim_{n\to\infty}\frac{4n^2+6n+1}{3n^2+2}-1=\frac{4}{3}-1=\frac{1}{3}<1\,. \tag{1}$$

By the *root test* the series is convergent.

(1.2) **[4p]** It follows that,

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}}=\lim_{n\to\infty}2\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}2\left(1+\frac{1}{n}\right)^{-n}=\frac{2}{e}<1\,. \tag{2}$$

By the *ratio test* the series converge.

(1.3) **[4p]** We can separate the series in two terms

$$\sum_{n=1}^{\infty}\frac{(3n)!}{(3n+1)!}+\sum_{n=1}^{\infty}\frac{4^{n+1}}{(3n+1)!}\,, \tag{3}$$

and show that while the second series is convergent, the first one is divergent. For the second one, it is enough to use the *ratio test*

$$\lim_{n\to\infty} \frac{\frac{4^{n+2}}{(3n+4)!}}{\frac{4^{n+1}}{(3n+1)!}} = \lim_{n\to\infty} \frac{4}{(3n+4)(3n+3)(3n+2)} = 0 < 1\,. \tag{4}$$

On the other hand the first series can be rewritten as

$$\sum_{n=1}^{\infty} \frac{1}{3n+1}\,, \tag{5}$$

which behaves as the harmonic series and hence diverge. A more formal proof can be given using the *comparison test*. Since our series is the sum of a convergent series and a divergent series, it follows that it is divergent.

(1.4) [4p] This series is divergent, to show it we can use the *comparison test* and compare it to the harmonic series

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^4-1}}{n^3}}{\frac{1}{n}} = 1 > 0\,, \tag{6}$$

and since the harmonic series is divergent, it follows that our series is also divergent.

(2) [16p] Convergence radius

Calculate the values of $x \in \mathbb{R}$ such that the following series are convergent:

(2.1) [4p] $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$,

(2.2) [4p] $\sum_{n=1}^{\infty} (-1)^n \frac{n\,(x-3)^{2n}}{(n+1)!}$,

(2.3) [4p] $\sum_{n=1}^{\infty} \frac{8^n\, x^{3n}}{3n(3n+3)}$,

(2.4) [4p] $\sum_{n=1}^{\infty} \frac{n^n}{n!} (x-1)^n$,

Solution:

(2.1) [4p] For this series we can use the *root test*

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x|}{2} < 1 \Longleftrightarrow -2 < x < 2\,. \tag{7}$$

We can also use that for $|p| < 1$ have

$$\sum_{n=1}^{\infty} p^n = \frac{1}{1-p}\,, \tag{8}$$

to show that the series converge for $|x| < 2$ and to find its limits in this range.

It remains to be seen what happens in the case where $|x| = 2$. In this case we have the series,

$$\sum_{n=1}^{\infty} 1\,, \tag{9}$$

which is clearly divergent, for instance we have that $\lim_{n\to\infty} a_n \neq 0$. As a result we conclude that the series is convergent only for $|x| < 2$.

(2.2) **[4p]** Here we apply the *ratio test*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)|x-3|^{2n+3}}{(n+2)!}}{\frac{n\,|x-3|^{2n}}{(n+1)!}} = \lim_{n\to\infty} \frac{|x-3|^3(n+1)}{n(n+2)} = 0 < 1\,, \tag{10}$$

and therefore the series is convergent $\forall\, x \in \mathbb{R}$.

(2.3) **[4p]** Once more, we use the *ratio test*,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{8^{n+1}\,|x|^{3n+3}}{(3n+3)(3n+6)}}{\frac{8^n\,|x|^{3n}}{3n(3n+3)}} = \lim_{n\to\infty} \frac{24n\,|x|^3}{3n+6} = 8\,|x|^3 < 1 \iff -\frac{1}{2} < x < \frac{1}{2}\,. \tag{11}$$

It remains to be shown what happens when $|x| = 1/2$. In this case the series reads

$$\sum_{n=1}^{\infty} \frac{1}{3n(3n+3)}\,. \tag{12}$$

We can use the *direct comparison test* with $c_n = 1/n^2$ to show that the series above is convergent. Indeed,

$$\sum_{n=1}^{N} \frac{1}{3n(3n+3)} \le \sum_{n=1}^{N} \frac{1}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}\,, \tag{13}$$

for all $N \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we conclude that the series in Eq. (12) is also convergent.

As a result, we find that the series is convergent for $|x| \le 1/2$.

(2.4) **[4p]** Again, using the *ratio test* we find

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}\,|x-1|^{n+1}}{\frac{n^n}{n!}\,|x-1|^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n |x-1| \\ &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n |x-1| = e\,|x-1| < 1 \iff 1-\frac{1}{e} < x < 1+\frac{1}{e}\,.\end{aligned} \tag{14}$$

We have to see what happens when $x = 1/e + 1$ and $x = 1 - 1/e$. In this case the series is given by

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} e^{-n}\,, \tag{15}$$

We apply the *direct comparison test* to show that the series above is divergent. Using the inequality in exercise (4.4) we have

$$\frac{n^n}{n!}e^{-n} \geq \frac{n^n}{e\,n\left(\frac{n}{e}\right)^n}e^{-n} = \frac{1}{e\,n} \qquad \forall n \in \mathbb{N}\,, \tag{16}$$

and since the series

$$\sum_{n=1}^{\infty} \frac{1}{e\,n}\,, \tag{17}$$

diverges, we conclude that the series in Eq. (15) also diverges.

Therefore, we find that the series of the exercise is convergent only when $|x| < \frac{1}{e} + 1$.

(3) [16p] Telescopic series

A telescopic sum is defined as

$$\sum_{n=1}^{N} (a_n - a_{n-1})\ , \tag{18}$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence. A telescopic series is a series whose partial sums are telescopic sums. Show whether the following series are telescopic and, if they converge, find their value:

(3.1) [4p] $\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$,

(3.2) [4p] $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$,

(3.3) [4p] $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$,

(3.4) [4p] $\sum_{n=1}^{\infty}(-1)^n$,

Solution:

(3.1) [4p] Given the definition, the series is clearly telescopic. We have that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{N+2}\right) = \frac{1}{2}\,. \tag{19}$$

(3.2) [4p] To show whether the series is telescopic, we need to work out the term in the sum

$$\frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\,, \tag{20}$$

We see that the series is telescopic, we have that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1\,. \tag{21}$$

(3.3) [4p] Again we start working out the term in the sum

$$\frac{1}{n^2-1} = \frac{1}{2}\frac{(n+1)-(n-1)}{(n+1)(n-1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}\right). \tag{22}$$

We see that the series is the sum of two telescopic series. We find that

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n^2-1} &= -\lim_{N\to\infty}\left[\frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{n} - \frac{1}{n-1}\right) + \frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{n+1} - \frac{1}{n}\right)\right] \\ &= -\frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{N} - 1 + \frac{1}{N+1} - \frac{1}{2}\right) = \frac{3}{4}\,. \end{aligned} \tag{23}$$

(3.4) [4p] This series is telescopic since we can rewrite the partial sum as

$$\sum_{n=1}^{N} (-1)^n = \frac{1}{2}\sum_{n=1}^{N}\left[(-1)^n - (-1)^{n-1}\right]\,. \tag{24}$$

However it is not convergent, as we can show by noting that the limit of the partial sums is not unique, i.e.

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n &= \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\left[(-1)^n - (-1)^{n-1}\right] = \frac{1}{2}\lim_{N\to\infty}\left[(-1)^N - 1\right] \\ &= \begin{cases} -1 & \text{if N odd} \\ 0 & \text{if N even} \end{cases}\,. \end{aligned} \tag{25}$$

(4) [12p] Euler's number and Stirling formula

Prove the following statements:

(4.1) [3p] The sequence $\left(e_n := \left(1 + \frac{1}{n}\right)^n\right)_{n\in\mathbb{N}^*}$ is monotonically increasing while the sequence $\left(a_n := \left(1 + \frac{1}{n}\right)^{n+1}\right)_{n\in\mathbb{N}^*}$ is monotonically decreasing.

(4.2) [3p] Given the following definition of the Euler's number:

$$e := \sum_{n=0}^{\infty} \frac{1}{n!}\,, \tag{26}$$

we have that $\lim_{n\to\infty} e_n = e$ and $\lim_{n\to\infty} a_n = e\,.$

Hint: The binomial theorem could be useful for the demonstration.

(4.3) [3p] The following equalities hold

$$\left(1+\frac{1}{1}\right)^1 \left(1+\frac{1}{2}\right)^2 \left(1+\frac{1}{3}\right)^3 \ldots \left(1+\frac{1}{n-1}\right)^{n-1} = \frac{n^n}{n!}\,, \tag{27}$$

$$\left(1+\frac{1}{1}\right)^2 \left(1+\frac{1}{2}\right)^3 \left(1+\frac{1}{3}\right)^4 \ldots \left(1+\frac{1}{n-1}\right)^{n} = \frac{n^n}{(n-1)!}\,. \tag{28}$$

(4.4) [3p] From the previous equality, the monotony of the sequences, and their limit to e, it follows that

$$e\left(\frac{n}{e}\right)^n \le n! \le e\,n\left(\frac{n}{e}\right)^n\,. \tag{29}$$

Note: More precisely, one can show that the value of $n!$ for large n can be approximated by the Stirling formula: $n! \simeq \sqrt{2\pi\,n}\,\left(\frac{n}{e}\right)^n$. [It is not required to prove this].

Solution:

(4.1) [3p] Let us begin with e_n. We have that $\forall\, n \in \mathbb{N}$

$$\begin{aligned}
\frac{e_{n+1}}{e_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} \\
&= \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}\left(1+\frac{1}{n}\right) \\
&= \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1}\left(1+\frac{1}{n}\right) \\
&= \left(1-\frac{1}{(n+1)^2}\right)^{n+1}\left(1+\frac{1}{n}\right) \\
&\ge \left(1-\frac{n+1}{(n+1)^2}\right)\left(1+\frac{1}{n}\right) \\
&= \left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right) \\
&= 1\,,
\end{aligned} \tag{30}$$

and therefore $e_{n+1} \ge e_n\ \forall\, n \in \mathbb{N}$. We conclude that the sequence is monotonically increasing.

Let us proceed now with a_n,

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} \\ &= \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^{n+2}\left(1+\frac{1}{n}\right)^{-1} \\ &= \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2}\left(1+\frac{1}{n}\right)^{-1} \\ &= \left(1+\frac{1}{n(n+2)}\right)^{n+2}\left(1+\frac{1}{n}\right)^{-1} \\ &\geq \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{-1} \\ &= 1\,,\end{aligned} \tag{31}$$

and therefore we have shown that $a_{n+1} \leq a_n \; \forall\, n \in \mathbb{N}$. We conclude that the sequence is monotonically decreasing. **Note:** In both derivations we used the Bernoulli's inequality, i.e.

$$(1+x)^n \geq 1 + nx\,, \tag{32}$$

valid for every $n \in \mathbb{N}$ and $x \geq -1$.

(4.2) [3p] Let us start with e_n. To get the limit we make use of the binomial formula and expand

$$\begin{aligned}e_n &= \left(1+\frac{1}{n}\right)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &= \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}\,\frac{n-1}{n}+\frac{1}{3!}\,\frac{(n-1)(n-2)}{n^2}+\cdots+\frac{1}{n!}\,\frac{n!}{n^n}\,,\end{aligned} \tag{33}$$

The coefficients multiplying $1/k!$ in each term tend to 1 as $n \to \infty$ for every term, and hence

$$\lim_{n\to\infty} e_n = \sum_{k=0}^{\infty}\frac{1}{k!}\,, \tag{34}$$

which is equal to e by definition. On the other hand for a_n we have

$$a_n = \left(1+\frac{1}{n}\right)e_n\,, \tag{35}$$

and therefore we can see it as the product of two sequences that converge to 1 and to e, respectively. By proposition 3.1.6 iii), we get

$$\lim_{n\to\infty} a_n = e\,. \tag{36}$$

(4.3) [3p] Simplifying each of the terms in the product we readily find

$$\left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n}{n-1}\right)^{n-1} = \frac{n^{n-1}}{(n-1)!} = \frac{n^n}{n!}\,, \tag{37}$$

where the numerator of the k^{th} term in the left-hand side cancel k powers of the denominator of the $(k+1)^{\text{th}}$ term. Similarly, for the other equation we have

$$\left(\frac{2}{1}\right)^2 \left(\frac{3}{2}\right)^3 \left(\frac{4}{3}\right)^4 \cdots \left(\frac{n}{n-1}\right)^{n} = \frac{n^n}{(n-1)!}\,, \tag{38}$$

(4.4) [3p] Let us start with the first equality. Using the definition of e_n, this can be simply written as

$$\prod_{i=1}^{n-1} e_n = \frac{n^n}{n!}\,, \tag{39}$$

Given that e_n is a monotonically increasing sequence, we have that each of its terms is smaller or equal than its limit, i.e. $e_k \leq e$. Hence,

$$\frac{n^n}{n!} = \prod_{i=1}^{n-1} e_n \leq e^{n-1} \Longrightarrow e\left(\frac{n}{e}\right)^n \leq n!\,. \tag{40}$$

We can proceed in a similar way with a_n, with the only difference that a_n is a monotonically decreasing sequence,

$$\frac{n^n}{(n-1)!} = \prod_{i=1}^{n-1} a_n \geq e^{n-1} \Longrightarrow n! \leq e\,n\left(\frac{n}{e}\right)^n\,. \tag{41}$$

Combining the two inequalities we arrive to the desired expression

$$e\left(\frac{n}{e}\right)^n \leq n! \leq e\,n\left(\frac{n}{e}\right)^n\,. \tag{42}$$