[MAT121] Analysis I
Homework 6

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Deadline: Friday, 03.11.2017, 12:00
Remember to write your name and your assistant's name

(1) [16p] Convergence of series

Show explicitly whether the following series are convergent or divergent:

(1.1) [4p] $\sum_{n=1}^{\infty} \left(\frac{4n^2+6n+1}{3n^2+2} - 1 \right)^n$,

(1.2) [4p] $\sum_{n=1}^{\infty} (-1)^n \frac{2^n\, n!}{n^n}$,

(1.3) [4p] $\sum_{n=1}^{\infty} \frac{(3n)! + 4^{n+1}}{(3n+1)!}$,

(1.4) [4p] $\sum_{n=1}^{\infty} \frac{\sqrt{n^4-1}}{n^3}$,

(2) [16p] Convergence radius

Calculate the values of $x \in \mathbb{R}$ such that the following series are convergent:

(2.1) [4p] $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$,

(2.2) [4p] $\sum_{n=1}^{\infty} (-1)^n \frac{n\,(x-3)^{2n}}{(n+1)!}$,

(2.3) [4p] $\sum_{n=1}^{\infty} \frac{8^n\, x^{3n}}{3n(3n+3)}$,

(2.4) [4p] $\sum_{n=1}^{\infty} \frac{n^n}{n!} (x-1)^n$,

(3) [16p] Telescopic series

A telescopic sum is defined as

$$\sum_{n=1}^{N} (a_n - a_{n-1}) \ , \qquad (1)$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence. A telescopic series is a series whose partial sums are telescopic sums. Show whether the following series are telescopic and, if they converge, find their value:

(3.1) **[4p]** $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$,

(3.2) **[4p]** $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$,

(3.3) **[4p]** $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$,

(3.4) **[4p]** $\sum_{n=1}^{\infty}(-1)^n$,

(4) **[12p]** Euler's number and Stirling formula

Prove the following statements:

(4.1) **[3p]** The sequence $\left(e_n := \left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}^*}$ is monotonically increasing while the sequence $\left(a_n := \left(1+\frac{1}{n}\right)^{n+1}\right)_{n\in\mathbb{N}^*}$ is monotonically decreasing.

(4.2) **[3p]** Given the following definition of the Euler's number:

$$e := \sum_{n=0}^{\infty}\frac{1}{n!}\,, \tag{2}$$

we have that $\lim_{n\to\infty} e_n = e$ and $\lim_{n\to\infty} a_n = e$.

Hint: The binomial theorem could be useful for the demonstration.

(4.3) **[3p]** The following equalities hold

$$\left(1+\frac{1}{1}\right)^1\left(1+\frac{1}{2}\right)^2\left(1+\frac{1}{3}\right)^3\cdots\left(1+\frac{1}{n-1}\right)^{n-1} = \frac{n^n}{n!}\,, \tag{3}$$

$$\left(1+\frac{1}{1}\right)^2\left(1+\frac{1}{2}\right)^3\left(1+\frac{1}{3}\right)^4\cdots\left(1+\frac{1}{n-1}\right)^{n} = \frac{n^n}{(n-1)!}\,. \tag{4}$$

(4.4) **[3p]** From the previous equality, the monotony of the sequences, and their limit to e, it follows that

$$e\left(\frac{n}{e}\right)^n \le n! \le e\,n\left(\frac{n}{e}\right)^n\,. \tag{5}$$

Note: More precisely, one can show that the value of $n!$ for large n can be approximated by the Stirling formula: $n! \simeq \sqrt{2\pi\,n}\left(\frac{n}{e}\right)^n$. [It is not required to prove this].