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Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Sheet 5

Deadline: 27.10.17, 12:00.

Let $\mathcal{G} := \{\tau_i : 0 \leq i \leq n\}$ be a partition of $\Omega = [a, b]$, i.e. $\tau_i := [x_i, x_{i+1}]$ for some $a = x_0 < x_1 < \ldots < x_{n+1} = b$. We define

$$\begin{aligned} S_{\mathcal{G}}^{1,0} &:= \left\{\varphi \in C^0([a,b]) \middle| \forall \tau \in \mathcal{G} : \varphi\big|_\tau \in \mathbb{P}_1\right\} \cap H_0^1(\Omega) \\ &= \operatorname{span}\{b_i(x), 1 \leq i \leq n\}, \end{aligned}$$

where

$$b_i(x) := \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in \tau_{i-1}, \\ \frac{x - x_{i+1}}{x_i - x_{i+1}} & x \in \tau_i, \\ 0 & \text{otherwise,} \end{cases}$$

are the piecewise linear "hat functions" on $\mathcal{G}$.

Exercise 1 (6 P.)

(a) Implement a (MATLAB) program, which computes for a given partition $\mathcal{G}$ and a given function $f : \Omega \to \mathbb{R} \in C^0(\Omega)$ the solution of the linear system $\mathbf{Au} = \mathbf{r}$, where

$$\mathbf{A} = (a_{ij})_{i,j=1}^n, \quad a_{ij} := \int_0^1 b_i' \cdot b_j' + \int_0^1 b_i \cdot b_j \qquad \text{and} \qquad \mathbf{r} = (r_i)_{i=1}^n, \quad r_i := \int_0^1 f \cdot b_i.$$

Find an explicit formula for the entries of $\mathbf{A}$. You can use the MATLAB function `integral` to compute the vector $\mathbf{r}$.

(b) Use your program to determine approximated solutions of the boundary value problem

$$-u''(x) + u(x) = 1 \quad \text{in } \Omega = (0,1), \qquad u(0) = u(1) = 0. \qquad \text{(BVP)}$$

Plot the exact solution of the problem (BVP) and the approximated solutions for different partitions of $\mathcal{G}$.

Exercise 2 (6 P.)

We set $\Omega = [0, 1]$. Let $\mathcal{G}$ a partition of $\overline{\Omega}$ and $\tilde{\mathcal{G}}$ a refinement of $\mathcal{G}$.

(a) Write a (MATLAB) program, that refines a given subdivision $\mathcal{G}$ by interval halving. The output is the resulting subdivision $\tilde{\mathcal{G}}$.

(b) Write a (Matlab-)program, that calculates for the given coefficient vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ the $H^1(\Omega)$-norm of the difference of the corresponding functions $u := \sum_{i=1}^n \mathbf{u}_i b_i \in S_{\mathcal{G}}^{1,0}$ and $v := \sum_{i=1}^n \mathbf{v}_i b_i \in S_{\mathcal{G}}^{1,0}$.

Hint: Use the matrix $\mathbf{A}$ from Exercise 1.

(c) Write a (Matlab-)program, that calculates for a given coefficient vector $\mathbf{u}$ of a function $u \in S^{1,0}_{\mathcal{G}}$ the coefficient vector of $\mathbf{u}$ with respect to the refined mesh $\tilde{\mathcal{G}}$ and the space $S^{1,0}_{\tilde{\mathcal{G}}}$ respectively.

(d) Write a (Matlab-)programm, that estimates the error of the finite-element-approximation $u \in S^{1,0}_{\mathcal{G}}$ from Exercise 1 in the $H^1(\Omega)$-norm. To do this, choose a suitable refinement $\tilde{\mathcal{G}}$ of the partition $\mathcal{G}$ (see (a)), determine the coefficients of u with respect to $\tilde{\mathcal{G}}$ (see (c)) and compare them with the finite-element-approximation with respect to the subdivision $\tilde{\mathcal{G}}$ (see (b)).

Exercise 3 (3. P.)
Consider the following partial differential equation for $f \in C^0([0,1])$:

$$u''(x) = f(x) \quad \text{in } \Omega = (0,1), \qquad u'(0) = u'(1) = 0. \tag{1}$$

(a) Explain why this equation is not well posed in $H^1(\Omega)$.

(b) Let $f(x) = (x - 1/2)$ and add the additional boundary condition $u(0) = 0$ to the PDE. What basis for a finite dimensional subspace $S \subset S^1_{\mathcal{G}} := \left\{\varphi \in C^0([0,1]) \middle| \forall \tau \in \mathcal{G} : \varphi\big|_\tau \in \mathbb{P}_1\right\}$ do you choose to approximate the solution u numerically ?