[MAT121] Analysis I
Homework 4

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Deadline: Friday, 20.10.2017, 12:00
Remember to write your name and your assistant's name

(1) [5p] Two equivalent notions of convergence and continuity

A sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is eventually in a set $X \subset \mathbb{R}$ if and only if $\{n \in \mathbb{N}, x_n \notin X\}$ is finite. Let $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a convergent sequence,

$$y = \lim_{n\to\infty} y_n \ .$$

Prove that $(y_n)_{n\in\mathbb{N}}$ is eventually in every open interval $(y-q, y+q)$, $q \in \mathbb{Q}^+$. Conversely, prove that if a sequence $(y_n)_{n\in\mathbb{N}}$ is eventually in every open interval $(y-\frac{1}{m}, y+\frac{1}{m})$, $m \in \mathbb{N}$ and $y \in \mathbb{R}$, then

$$\lim_{n\to\infty} y_n = y \ .$$

(2) [10p] Limit of sequences

(2.1) [3p] Find the limit of $\sqrt{n+1} - \sqrt{n}$ as $n \to \infty$.

(2.2) [4p] Using the definition of convergence, prove that $1 - \frac{1}{n+\frac{2}{n}}$ converges to 1 as $n \to \infty$.

(2.3) [3p] Find the limit of $n - \frac{1}{\cos(n)\left(1+\tan(n)^2\right)}$ as $n \to \infty$.

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(*Hint*: the formula $\sin(n)^2 + \cos(n)^2 = 1$ may be useful)

(3) [8p] A property of lim sup and lim inf

Let $\mathrm{Cl}(x_n, n \in \mathbb{N}) \subset \mathbb{R}$ denote the set of cluster points (Häufungspunkt) of a *bounded* sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$. Prove that

$$\limsup_{n\to\infty} x_n = \max\big\{x \in \mathrm{Cl}(x_n, n \in \mathbb{N})\big\} \ ,$$
$$\liminf_{n\to\infty} x_n = \min\big\{x \in \mathrm{Cl}(x_n, n \in \mathbb{N})\big\} \ .$$

(4) [8p] Manipulation of non-convergent sequences

Give at least one explicit example of the following two situations:

1. **[4p]** A bounded sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ that is not convergent, but whose square $(x_n^2)_{n\in\mathbb{N}}$ is convergent.

2. **[4p]** Two bounded sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ that are not convergent, but whose sum $(x_n + y_n)_{n\in\mathbb{N}}$ is convergent.

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Remark: You are expected to give the proof that the explicit sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ provided by you are not convergent (*e.g.* showing that their lim sup and lim inf do not coincide), and that $(x_n^2)_{n\in\mathbb{N}}$ and $(x_n + y_n)_{n\in\mathbb{N}}$ are instead convergent.

(5) [12p] Recursively defined sequences

(5.1) [3p] Find the values of $\xi \in \mathbb{R}$ for which the following recursively defined sequence $(x_n)_{n\in\mathbb{N}}$ converges:

$$x_0 = \xi$$
$$x_n = x_{n-1}^n \ , \ \forall n \geq 1 \ .$$

(5.2) [4p] Let $a \in \mathbb{R}^+$ be fixed. Study the convergence (and find the limit if it converges) of the sequence $(x_n)_{n\in\mathbb{N}}$, defined recursively as

$$x_0 = a$$
$$x_n = a - \frac{1}{2}x_{n-1} \ , \ \forall n \geq 1 \ .$$

(5.3) [5p] Let $a \in \mathbb{R}^+$ be fixed, and let $\xi > a$. Let the sequence $(x_n)_{n\in\mathbb{N}}$ be recursively defined as

$$x_0 = \xi$$
$$x_n = \frac{1}{x_{n-1}}(2ax_{n-1} - a^2) \ , \ \forall n \geq 1 \ .$$

Prove that $(x_n)_{n\in\mathbb{N}}$ converges, and find its limit showing that it *does not* depend on ξ.

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(*Hint*: You may use without proof that if $x_n \to x$ (the sequence converges), then the limit point x satisfies: $x = a - \frac{1}{2}x$ in **(5.2)**; and $x = \frac{1}{x}(2ax - a^2)$ in **(5.3)**)

(6) [5p] Infinitely many cluster points

Write a real-valued sequence with infinitely many different cluster points.

(7) [12p] The diagonal trick

Let $(f_n)_{n\in\mathbb{N}}$ be a family of functions, with $f_n : \mathbb{N} \to \mathbb{R}$ for each $n \in \mathbb{N}$. Suppose that, in addition, for any $a \in \mathbb{N}$, $\big(f_n(a)\big)_{n\in\mathbb{N}}$ is a bounded sequence. Then there exists a subsequence $(f_{n(i)})_{i\in\mathbb{N}}$ such that $\big(f_{n(i)}(a)\big)_{i\in\mathbb{N}}$ converges for all $a \in \mathbb{N}$.

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(*Hint*: Use Bolzano-Weierstrass theorem to extract a subsequence $(f_{n_1(i)})_{i\in\mathbb{N}}$ such that $f_{n_1(i)}(1)$ converges as $i \to \infty$. Now extract from $(f_{n_1(i)})_{i\in\mathbb{N}}$ a subsequence $(f_{n_2(i)})_{i\in\mathbb{N}}$ such that also $f_{n_2(i)}(2)$ converges, etc. Now we have a sequence $(f_{n_a(i)})_{i\in\mathbb{N}}$ such that $f_{n_a(i)}(b)$ converges for all $b \leq a$ as $i \to \infty$. The diagonal trick is to choose the common subsequence $(f_{n(i)})_{i\in\mathbb{N}}$ by setting $n(i) = n_i(i)$, and take $i \to \infty$. Why does this allows convergence for any $a \in \mathbb{N}$?)