computational mathematics

Prof. Dr. S. Sauter
Institut für Mathematik
Universität Zürich

Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Sheet 3

Deadline: 13.10.17, 12:00.

Exercise 1 (4 P.)
Let $\phi \in L^2(\Omega)$ and u_ϕ the solution of the boundary value problem

$$-u_\phi'' + cu_\phi' + u_\phi = \phi \quad \text{in } \Omega, \qquad u_\phi(0) = u_\phi(1) = 0.$$

a) Let $c = 0$. Show that there exists a constant $C > 0$ such that: $|u_\phi|_{H^2(\Omega)} \leq C \|\phi\|_{L^2(\Omega)}$.

b) Let $c = 1$. Show that there exists a constant $C > 0$ such that: $|u_\phi|_{H^2(\Omega)} \leq C \|\phi\|_{L^2(\Omega)}$.

Hint for a): Get first an inequality for $\|\cdot\|_{H^1(\Omega)}$ instead of $|\cdot|_{H^2(\Omega)}$, by multiplying the differential equation by u_ϕ and integrating over Ω. Use partial integration.
Hint for b): Show that

$$\int_\Omega u_\phi'(x) u_\phi(x) dx = 0,$$

and conclude with a).

Exercise 2 (6 P.)
Let X, Y be Banach spaces, and let $X \overset{c}{\hookrightarrow} Y$ (which implies $Y' \overset{c}{\hookrightarrow} X'$). Let the bilinear form

$$a_0 : X \times X \to \mathbb{R}$$

be continuous, symmetric and coercive. Moreover let the bilinear form a_1

$$a_1 : X \times Y \to \mathbb{R}$$

satisfy

$$a_1(u, v) \leq C\|u\|_X \|v\|_Y, \quad \forall u \in X, v \in Y.$$

The goal is to show that the linear mapping $K : X \to X$, $w \to Kw$ defined by

$$a_0(Kw, v) = a_1(w, v), \quad \forall v \in X$$

is compact.

(a) Let X, Y, Z be Banach spaces. Let $K \in \mathcal{L}(X, Y)$ be a compact operator. Show that for $T_1 \in \mathcal{L}(Y, Z)$ and $T_2 \in \mathcal{L}(Z, X)$ the operators $T_1 \circ K \in \mathcal{L}(X, Z)$ and $K \circ T_2 \in \mathcal{L}(Z, Y)$ are compact.

(b) Let $w \in X$. Show that the operator

$$A_w : X \to \mathbb{R}, \; v \to a_1(w, v)$$

is a compact operator in X' (use that $Y' \overset{c}{\hookrightarrow} X'$).

(c) For $\ell \in X'$, consider the problem: Find $u_\ell \in X$ s.t.

$$a_0(u_\ell, v) = \ell(v) \quad \forall v \in X.$$

Define by S the solution operator $S : X' \to X, \ell \to u_\ell$ given by the Lax-Milgram Theorem (Satz 3.2). Use coercivity of $a_0(\cdot, \cdot)$ and continuity of ℓ to prove $S \in \mathcal{L}(X', X)$.

(d) Derive the compactness of K using (a)-(c).

Exercise 3 (2 P.)
Let X be an infinite dimensional Hilbert space and let x_n be a orthonormal sequence in X, i.e.

$$x_n \in X, \quad \|x_n\|_X = 1,$$
$$\langle x_n, x_m \rangle_X = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

Prove by contradiction that there exist no convergent subsequence of $(x_n)_{n=1}^\infty$ in X.