EXERCISES. XIII

A- The goal of this exercise is to give examples of universally open morphisms which are not flat.

(1) Show that $f : Y \to X$ is universally open iff $f_{red} : Y_{red} \to X_{red}$ is universally open. Use this fact to construct an non-flat morphism which is universally open. (*Hint:* Let k be a field, $A = k[t]$ and $B = k[t] \oplus k$ where the multiplication of B is given by $(P, a)(Q, b) = (PQ, P(0)b + Q(0)a)$ for (P, a) and (Q, b) in B. Show that B is an A-algebra and that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is universally open but not flat.)

(2) In the rest of the exercise we will show that non-flat morphism between reduced schemes can also be universally open. Let k be a field (of characteristic different from 2), $A = k[x, y]$ and set $X = \mathrm{Spec}(A)$. Let $G = \mathbb{Z}/2\mathbb{Z}$ be the cyclic group of order 2 acting on A by the rule $\tau(x) = -x$ and $\tau(y) = -y$ (where τ denote the non-zero element of G). Let A^G be the ring of fixed elements of A and $X/G = \mathrm{Spec}(A^G)$.

  (a) Show that A^G is the subring of A generated by x^2, y^2 and xy.

  (b) Show that G acts transitively on each fiber of the morphism $X \to X/G$.

  (c) Show that A is not a flat A^G-algebra. (*Hint:* By contradiction. If A was flat over A^G show that $A/Ax^2 + Axy \simeq k[x, y]/(x^2, xy)$ would be flat over $A^G/A^Gx^2 + A^Gxy \simeq k[y^2]$. Show that this is impossible.)

  (d) We want to show that $X \to X/G$ is a universally open morphism. Use the valuative critera to reduce to show the following: for any d.v.r. R and any morphism $\mathrm{Spec}(R) \to X/G$, the morphism $p : X \times_{X/G} \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is open. Assume by contradiction that the morphism p is not finite. Show that there exists $x \in p^{-1}(o)$ which is open in $X \times_{X/G} \mathrm{Spec}(R)$. Using question (b) deduce that $p^{-1}(o)$ is an open subset of $X \times_{X/G} \mathrm{Spec}(R)$. Show that this is absurde.