Exercises. XV

A- Let k be a field and $n \in \mathbb{N}$. We identify $\mathbb{P}^n_k$ with $\mathrm{Proj}(k[t_0, \dots, t_n])$.

(1) Sow that $\mathbb{P}^n_k$ is a smooth k-scheme of dimension n.

(2) For $i, j \in [\![0, n]\!]$, show that there exists a global vector field (i.e., a derivation) $\nu_{i,j} : \mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_{\mathbb{P}^n_k}$ sending a homogenous function $f \in \mathcal{O}_{\mathbb{P}^n_k}(U)$ defined over an open subset $U \subset \mathbb{P}^n_k$ to $\nu_i(f) = t_i \dfrac{\partial f}{\partial t_j}$.

(3) Show that the $\nu_{i,j}$ generate the quasi-coherent $\mathcal{O}_{\mathbb{P}^n_k}$-module $\mathcal{T}_{\mathbb{P}^n_k} = \mathrm{Der}_k(\mathcal{O}_{\mathbb{P}^n_k}, \mathcal{O}_{\mathbb{P}^n_k})$. Show that we have the relation $\sum_{i=0}^n \nu_{i,i} = 0$.

(4) Show that for $j \in [\![0, n]\!]$, we have a morphism of $\mathcal{O}_{\mathbb{P}^n_k}$-modules $\alpha_j : \mathcal{O}_{\mathbb{P}^n_k}(1) \to \mathcal{T}_{\mathbb{P}^n_k}$ sending t_i to $\nu_{i,j}$.

(5) Show that we have an exact sequence of $\mathcal{O}_{\mathbb{P}^n_k}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^n_k} \to \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_k}(1) \xrightarrow{\sum_{i=0}^n \alpha_i} \mathcal{T}_{\mathbb{P}^n_k} \to 0.$$

(6) Deduce that we have a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n_k/k} \to \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_k}(-1) \to \mathcal{O}_{\mathbb{P}^n_k} \to 0.$$

(7) Deduce that $\Omega_{\mathbb{P}^1_k/k} \simeq \mathcal{O}_{\mathbb{P}^1_k}(-2)$.