INTRODUCTION TO ALGEBRAIC GEOMETRY

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Contents

1. Preliminaries on Sheaves and Locally Ringed Spaces

1.1. Topological spaces.

Recall that a topology on a set X is a subset $\mathcal{T} \subset \mathcal{P}(X)$ satisfying the following conditions

(0) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(1) If A and B are in $\mathcal{T}$ then so it $A \cap B$,
(2) If $(A_i)_{\in I}$ is a family of elements in $\mathcal{T}$, then $\bigcup_{i \in I} A_i$ is also in $\mathcal{T}$.

A topological space is a couple $(X, \mathcal{T})$ consisting of a set X endowed with a topology $\mathcal{T}$. When the topology $\mathcal{T}$ is understood, we write simply X. The elements of $\mathcal{T}$ are called the open subsets of the topological space $(X, \mathcal{T})$. We say that $Z \subset X$ is closed if $X - Z$ is open. $\emptyset$ and X are closed subsets. Arbitrary intersections and finite unions take closed subsets to closed subsets.

An application $f : X \to Y$ between topological spaces is continuous if $f^{-1}(U)$ is open in X whenever $U \subset Y$ is open. Such an application is called a continuous map of topological spaces. The set of continuous maps from X to Y is denoted by $\mathcal{C}^0(X, Y)$. We often denote the set $\mathcal{C}^0(X, \mathbb{R})$ by $\mathcal{C}^0(X)$. This is the ring of continuous real functions on X. We say that $f : X \to Y$ is a homeomorphism if f is continuous, bijective and its inverse f^{-1} is also continuous.

Example 1.1 —
0- Any set X can be endowed with the coarse topology $\mathcal{T}_{co} = \{\emptyset, X\}$ and the discrete topology $\mathcal{T}_{discrete} = \mathcal{P}(X)$.
1- Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset of X. A subset $V \subset Y$ is called open of there exists an open subset $U \subset X$ such that $V = Y \cap U$. This defines the induced topology $\mathcal{T}_{|Y}$ on Y. More generally, given an application $f : X' \longrightarrow X$ we define a topology $f^{-1}(\mathcal{T})$ on X' whose elements are the $f^{-1}(U)$ for $U \in \mathcal{T}$.
2- A subset $\Omega \subset \mathbb{R}^n$ is open if for every $x \in \Omega$, there exists $\epsilon > 0$ such that $B(x, \epsilon) = \{x' \in \mathbb{R}^n ; \|x - x'\| < \epsilon\}$ is contained in Ω. This defines the classical topology on $\mathbb{R}^n$. More generally, given a set X and a metric $\mathrm{d} : X \times X \to \mathbb{R}_+$, there exists a topology on X for which the opens are the subsets $\Omega \subset X$ such that whenever $x \in \Omega$ there exists $\epsilon > 0$ sufficiently small so that $B(x, \epsilon) \subset \Omega$.
3- Recall that a topological manifold M is a topological space such that for every $x \in M$ there exists an open subset $U \subset M$ containing x which is homeomorphic to an open subset of $\mathbb{R}^d$ (where d might depends on x).

1.2. Presheaves on topological spaces.

Definition 1.2 — *Let $(X, \mathcal{T})$ be a topological space. A* presheaf *F of sets (resp. groups, rings, modules, etc) on X consists of the following data:*

(1) *For every open set $U \subset X$, a set (resp. group, ring, module, etc) $F(U)$.*
(2) *For every inclusion $U \subset V$ of open subsets of X, an application (resp. a morphism of groups, rings, modules, etc) $\rho_U^V : F(V) \to F(U)$ such that $\rho_U^U = \mathrm{id}_{F(U)}$ and $\rho_U^V \circ \rho_V^W = \rho_U^W$ for any chain of inclusions $U \subset V \subset W$.*

The ρ_U^V *are called* restriction morphisms *(from* V *to* U*). The elements of* $F(U)$ *are called* sections *of* F *on* U.

Given a presheaf F on a topological space X and a section $s \in F(V)$ over an open set $V \subset X$, we usually write $s_{|U}$ for the restriction $\rho_U^V(s)$ of s to $U \subset V$.

Example 1.3 —

1- Let X be a topological space and A be a set (resp. group, ring, module, etc). The constant presheaf A_{cst} on X sends an open subset $U \subset X$ to $A_{cst}(U) = A$ and an inclusion $U \subset V$ to the identity of A.

2- Let X be a topological space. The presheaf $\mathcal{C}_X^0$ sends an open subset $U \subset X$ to the ring $\mathcal{C}^0(U)$ of continuous functions on U. Given an inclusion of open subsets $U \subset V$, $\rho_U^V : \mathcal{C}^0(V) \to \mathcal{C}^0(U)$ is the usual restriction of functions, which is clearly a morphism of rings. In particular, $\mathcal{C}_X^0$ is a sheaf of commutative rings on X.

3- Let X and Y be two topological spaces. On defines a presheaf of sets $\mathcal{C}_X^0(Y)$ by $\mathcal{C}_X^0(Y)(U) = \mathcal{C}^0(U, Y)$.

4- Recall that a $\mathcal{C}^n$-structure on a topological manifold M is a maximal family $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ where, $U_\alpha \subset M$ are opens, $\phi_\alpha : U \to \mathbb{R}^{d_\alpha}$ are continuous maps such that:

- Every point of M is contained in some U_α,
- $\phi_\alpha(U)$ is an open subset of $\mathbb{R}^{d_\alpha}$ and $U \to \phi_\alpha(U)$ is a homeomorphism,
- for α, $\beta \in A$, the composition:

$$\phi_\alpha(U_\alpha \cap U_\beta) \simeq U_\alpha \cap U_\beta \simeq \phi_\beta(U_\alpha \cap U_\beta)$$

  is a $\mathcal{C}^n$-diffeomorphism of open subsets of $\mathbb{R}^{d_\alpha}$ and $\mathbb{R}^{d_\beta}$.

Such a family is called an $\mathcal{C}^n$-atlas. Given a $\mathcal{C}^n$-structure on a topological manifold X, one can define a presheaf of commutative rings $\mathcal{C}_X^n$ on X by sending an open subset $U \subset X$ to the subring of $\mathcal{C}^0(U)$ consisting of those functions f such that whenever $U_\alpha \subset U$, the composition

$$\phi_\alpha(U_\alpha) \simeq U_\alpha \xrightarrow{f_{|U_\alpha}} \mathbb{R}$$

is a $\mathcal{C}^n$ function on the open subset $\phi_\alpha(U_\alpha) \subset \mathbb{R}^{d_\alpha}$. In the same way, one can define the sheaf of holomorphic functions on a complex analytic manifold.

Let X be a topological space and F and G two presheaves of sets (resp. groups, rings, modules etc) on X. A morphism of presheaves $f : F \to G$ is a family of applications (resp. morphisms of groups, rings, modules etc) $f_U : F(U) \to G(U)$ making the square commutative

$$\begin{array}{ccc} F(V) & \xrightarrow{f_V} & G(V) \\ \downarrow{\scriptstyle \rho_U^V} & & \downarrow{\scriptstyle \rho_U^V} \\ F(U) & \xrightarrow{f_U} & G(U) \end{array}$$

for every inclusion of opens $U \subset V$.

1.3. Germs of sections in a presheaf.

Let X be a topological space and $Y \subset X$. We denote $\mathscr{V}(Y)$ the set of opens in X that contains Y. These opens are called the *open neighborhoods* of Y. This an ordered set for the inclusion. If U, $V \in \mathscr{V}(Y)$, then also $U \cap V \in \mathscr{V}(Y)$. In particular, any finite family of elements in $\mathscr{V}(Y)$ is bounded below by some element of $\mathscr{V}(Y)$, namely their intersection.

Definition 1.4 — *Let X be a topological space and F a presheaf of sets (groups, rings, modules, etc) on X. Let $Y \subset X$ be any subset. Consider the set of couples (s, V) with $V \in \mathscr{V}(Y)$ and $s \in F(V)$. Two couples (s_1, V_1) and (s_2, V_2) are said to be equivalent (and we write $(s_1, V_1) \sim (s_2, V_2)$) if there exists $V \in \mathscr{V}(Y)$ contained in $V_1 \cap V_2$ and such that $(s_1)_{|V} = (s_2)_{|V}$. One checks immediately that this is an equivalence relation. A* germ *of sections of F on Y is an equivalence class for this equivalence relation. We denote $[s, V]$ (or $[s, V]_Y$ if we need to be more precise) the germ associated to a couple (s, V). The set of germs on Y is denoted $F(Y)$ (check that this is compatible with previous notation when Y is open). When $Y = \{x\}$ consists of one point, we write s_x and F_x for $[s, V]_{\{x\}}$ and $F(\{x\})$. The F_x are called the* stalks *or the* fibers *of F.*

Remark 1.5 — When F is a presheaf of groups, (resp. rings, modules), the set of germs $F(Y)$ for a subset $Y \subset X$ is naturally a group, (resp. ring, module). Indeed, given two germs $[s, V]$ and $[s', V']$ one can perform their product as follows

$$[s, V].[s', V'] = [s_{|V \cap V'}.s'_{|V \cap V'}, V \cap V'].$$

(Check that is well defined.) A similar formula work also for addition, etc.

Definition 1.4 gives a coherent way to extend any presheaf to all subsets of a topological spaces. In other terms, given a presheaf F on a topological space X, one can associate a presheaf $\tilde{F}$ on the discrete topological space $X_{discrete}$ having the same underlying set as X. The presheaf $\tilde{F}$ associate to a subset $Y \subset X$ the set of germs $\tilde{F}(Y)$ which was also denoted by $F(Y)$ in Definition 1.4. Given a morphism $f : F \to G$ of presheaves on X, there for every subset $Y \subset X$ a morphism $f_Y : \tilde{F}(Y) \to \tilde{G}(Y)$ which takes a germ $[s, V]$ to the germ $[f_V(s), V]$ (check that this well-defined!). These morphisms, are compatible with the restriction maps for inclusion of subsets $Z \subset Y$. Thus, they define a morphism $\tilde{f} : \tilde{F} \to \tilde{G}$ of presheaves.

Lemma 1.6 — *Let X be a topological space and denote $X_{discrete}$ the discrete topological space on the set of points of X. Let F be a presheaf on X and G a presheaf on $X_{discrete}$. Let $\tilde{F}$ be the presheaf of germs associated to F and G' be the presheaf on X obtained by restricting G to the open subsets of X. Then there exists a canonical isomorphism*

$$\hom(\tilde{F}, G) \simeq \hom(F, G').$$

Proof. Let $a : \tilde{F} \to G$ be a morphism of presheaves on $X_{discrete}$. As the restriction of $\tilde{F}$ to opens in X gives back the presheaf F, on get morphisms $F(U) = \tilde{F}(U) \xrightarrow{a_U} G(U)$ for all opens $U \subset X$. These morphisms defines a morphism of presheaves $a' : F \to G'$. We get in this way an application

$$\theta : \hom(\tilde{F}, G) \to \hom(F, G').$$

Let us prove that this application is injective. Suppose given two morphisms a_1, $a_2 : \tilde{F} \to G$ such that $\theta(a_1) = \theta(a_2)$. This means that for any open $U \subset X$, the morphisms

$$(a_1)_U,\ (a_2)_U : F(U) = \tilde{F}(U) \to G(U)$$

are equals. Now, let $Y \subset X$ be any subset and $[s,V] \in \tilde{F}(Y)$ a germ over Y. As a_1 and a_2 commute with restrictions, we should have $(a_i)_Y([s,V]) = [(a_i)_V(s), V]$ for $i \in \{1,2\}$. As, $(a_1)_V(s) = (a_2)_V(s)$, we get that $(a_1)_Y = (a_2)_Y$ for all $Y \subset X$.

To finish the proof, we still need to establish the surjectivity of θ. We fix a morphism of presheaves $b : F \to G'$. We may define a morphism $a : \tilde{F} \to G$ as follows. For $Y \subset X$ and $[s,V] \in F(Y)$ a germ, the restriction of $b(s) \in G(V)$ to Y is a well defined element of $G(Y)$. This gives application $\tilde{F}(Y) \to G(Y)$ which clearly commute with restriction maps. This gives our a. By construction, we have $\theta(a) = b$. □

1.4. **The direct and inverse images of presheaves.**

Definition 1.7 — *Let $f : Y \to X$ be a continuous map of topological spaces. Let F be a presheaf on Y. We define the presheaf f_*F on X by $(f_*F)(U) = F(f^{-1}(U))$ for any open $U \subset X$. This is the* direct image *of F along f.*

Let G be a presheaf on X. We define the presheaf f^*G by $(f^*G)(V) = G(f(V))$, the set of germs of G on the set $f(V)$ (which is not open in general). This is the inverse image *of F along f.*

Lemma 1.8 — *Let $g : Z \to Y$ and $f : Y \to X$ be two composable continuous maps of topological spaces. Given a presheaf G on Z, one has $f_*(g_*(G)) = (f \circ g)_*(G)$. Given a presheaf F on X, there is a canonical isomorphism $(f \circ g)^*(F) \simeq g^*(f^*(F))$.*

Proof. We clearly have $g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$ for any open subset $U \subset X$. If follows that $f_*(g_*(G))(U) = (g_*(G))(f^{-1}(U)) = G(g^{-1}(f^{-1}(U))) = G((f \circ g)^{-1}(U)) = (f \circ g)_*(F)(U)$.

We turn to the case of inverse images. For a subset $T \subset Z$, $g^*(f^*(F))(T)$ is the set of germs of $f^*(F)$ on the subset $g(T)$. Such a germ is given by an equivalence class of couples (t,V) where $t \in f^*(F)(V)$ for V an open neighborhood of $g(T)$ in Y. But t itself is an equivalence class of couples (s,U) where $s \in F(U)$ for U an open neighborhood of $f(V)$ in X. In other words, we may write an element of $g^*(f^*(F))(T)$ in the form $[t,V]_{g(T)} = [[s,U]_{f(V)}, V]_{g(T)}$.

On the other hand, $(f \circ g)^*(F)(T)$ is the set of germs of F on $f \circ g(T) = f(g(T))$. As U is an open neighborhood of $f(g(T))$, we may consider the germ $[s,U]_{f\circ g(T)}$. This gives an application $g^*(f^*(F))(T) \to (f \circ g)^*(F)(T)$. To see that this a bijection, we describe an inverse. Let $[s',U']_{f\circ g(T)} \in F(f \circ g(T))$. The subsets $f^{-1}(U')$ and $(f \circ g)^{-1}(U')$ are open neighborhoods of $g(T)$ and T. Thus, we may consider the element $[[s',U']_{f(f^{-1}(U'))}, f^{-1}(U')]_T \in g^*(f^*(F))(T)$. It is easily checked that this gives indeed an inverse to our previous map. □

Remark 1.9 — Let X be a topological space and denote $X_{discrete}$ the discrete topological space on the set of points of X. The identity of X is a continuous map $\iota : X_{discrete} \to X$. Let F be a presheaf on X. The presheaf ι^*F is simply the presheaf of germs on $X_{discrete}$ denoted by $\tilde{F}$ in Lemma 1.6. Moreover, if G is a presheaf on $X_{discrete}$ then ι_*G is the presheaf G' obtained by restricting G to opens of X. Lemma 1.6 states that there is a canonical isomorphism

$$\hom(\iota^*F, G) \simeq \hom(F, \iota_*G).$$

In the next proposition, we will see that this fact generalizes to any continuous map of topological spaces.

Proposition 1.10 — *With the notation of Definition 1.7, we have a canonical bijection*

$$\hom(G, f_*F) \simeq \hom(f^*G, F).$$

Proof. We break the proof in three steps. The first one, is a reduction to the case where X and Y are discrete topological spaces.
Step 1: Consider the following commutative square of continuous maps

$$\begin{array}{ccc} Y_{discrete} & \xrightarrow{f_{discrete}} & X_{discrete} \\ {\scriptstyle \iota_Y}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\ Y & \xrightarrow{\quad f \quad} & X \end{array}$$

where ι_X and ι_Y are the identity on the sets of points and $f_{discrete}$ coïncide with f on the sets of points. Using Lemma 1.8, we get a canonical isomorphism

$$\iota_Y^*(f^*(F)) \simeq (f_{discrete})^*(\iota_Y^*(F)).$$

Using Lemma 1.6, we now get natural isomorphisms

$$\hom(f^*(F), G) \simeq \hom(\iota_Y^*(f^*(F)), \iota_Y^*(G)) \simeq \hom((f_{discrete})^*(\iota_X^*(F)), \iota_Y^*(G)).$$

On the other hand, we also have by Lemma 1.6,

$$\hom(\iota_X^*(F), (f_{discrete})_*(\iota_Y^*(G))) \simeq \hom(F, f_*G)$$

using the fact that the restrictions of $\iota_X^*(F)$ and $(f_{discrete})_*(\iota_Y^*(G))$ to open subsets of X are simply F and $f_*(G)$. Thus, we are left to construct a natural isomorphism

$$\hom((f_{discrete})^*(\iota_X^*(F)), \iota_Y^*(G)) \simeq \hom(\iota_X^*(F), (f_{discrete})_*(\iota_Y^*(G))).$$

In other words, we may assume that X and Y are discrete.
Step 2: From now on, we suppose that X and Y are discrete topological spaces. Let $a : F \to f_*G$ be a morphism of presheaves on X. We associate to a a morphism $\theta_1(a) : f^*F \to G$ given by the compositions

$$\theta_1(a)_V : f^*F(V) = F(f(V)) \overset{a_{f(V)}}{\longrightarrow} G(f^{-1}f(V)) \xrightarrow{\rho_V^{f^{-1}f(V)}} G(V)$$

for all subsets $V \subset Y$. This defines a map

$$\theta_1 : \hom(F, f_*G) \to \hom(f^*F, G).$$

On the other hand, let $b : f^*F \to G$ be a morphism of presheaves on Y. We associate to b a morphism $\theta_2(b) : F \to f_*G$ given by the compositions

$$\theta_2(b)_U : F(U) \xrightarrow{\rho^U_{f(f^{-1}(U))}} F(f(f^{-1}(U))) \xrightarrow{b_{f^{-1}(U)}} G(f^{-1}(U)) = (f_*G)(U)$$

for all subsets $U \subset X$. This defines a map

$$\theta_2 : \hom(f^*F, G) \to \hom(F, f_*G).$$

Step 3: In this step we finish the proof of the proposition by showing that $\theta_1 \circ \theta_2$ and $\theta_2 \circ \theta_1$ are the identity.

Let $a : F \to f_*G$ and let us compute $\theta_2(\theta_1(a))$. Over a subset $U \subset X$, it is given by

$$F(U) \xrightarrow{\rho^U_{f(f^{-1}(U))}} F(f(f^{-1}(U))) \xrightarrow{\theta_1(a)_{f^{-1}(U)}} G(f^{-1}(U))\ .$$

But $\theta_1(a)_{f^{-1}(U)}$ is given by the composition

$$F(f(f^{-1}(U))) \xrightarrow{a_{f(f^{-1}(U))}} G(f^{-1}f(f^{-1}(U))) \xrightarrow{\rho_V^{f^{-1}f(f^{-1}(U))}} G(f^{-1}(U))\ .$$

As $f^{-1}(f(f^{-1}(U))) = f^{-1}(U)$, we get a commutative diagram

$$\begin{array}{ccccc}
F(U) & \xrightarrow{\rho^U_{f(f^{-1}(U))}} & F(f(f^{-1}(U))) & \xrightarrow{\theta_1(a)_{f^{-1}(U)}} & \\
\downarrow{\scriptstyle a_U} & & \downarrow{\scriptstyle a_{f(f^{-1}(U))}} & & \searrow \\
G(f^{-1}(U)) & = & G(f^{-1}f(f^{-1}(U))) & = & G(f^{-1}(U)).
\end{array}$$

This shows that $\theta_2(\theta_1(a)) = a$.

Now, let $b : f^*F \to G$ and let us compute $\theta_1(\theta_2(b))$. Over a subset $V \subset Y$, it is given by

$$F(f(V)) \xrightarrow{\theta_2(b)_{f(V)}} G(f^{-1}f(V)) \xrightarrow{\rho_V^{f^{-1}f(V)}} G(V)\ .$$

But, $\theta_2(b)_{f(V)}$ is given by the composition

$$F(f(V)) \xrightarrow{\rho^U_{f(f^{-1}(f(V)))}} F(f(f^{-1}(f(V)))) \xrightarrow{b_{f^{-1}(f(V))}} G(f^{-1}(f(V)))\ .$$

As $f(V) = f(f^{-1}(f(V)))$, we get a commutative diagram

$$\begin{array}{ccccc}
F(f(V)) & = & F(f(f^{-1}(f(V)))) & = & F(f(V)) \\
 & \searrow{\scriptstyle \theta_2(b)_{f(V)}} & \downarrow{\scriptstyle b_{f^{-1}(f(V))}} & & \downarrow{\scriptstyle b_V} \\
 & & G(f^{-1}(f(V))) & \xrightarrow[\rho_V^{f^{-1}(f(V))}]{} & G(V)
\end{array}$$

This shows that $\theta_1(\theta_2(b)) = b$. The proof of the proposition is complete. □

1.5. **Sheaves on a topological space.**

Definition 1.11 — *Let X be a topological space and F be a presheaf of sets (groups, rings, modules, etc) on X.*

1- *We say that F is* separated *if for any open $U \subset X$, any sections $s, t \in F(U)$ and any open covering $(U_i)_{i\in I}$ of U, the implication*

$$\big(\forall i \in I,\ s_{|U_i} = t_{|U_i}\big) \qquad \Rightarrow \qquad s = t$$

is true.

2- *We say that F is a* sheaf *if it is separated and satisfies the following property. Let $U \subset X$ be an an open subset and $(U_i)_{i\in I}$ be an open covering of X. Suppose we are given for each $i \in I$ a section $s_i \in F(U_i)$ such that $(s_i)_{|U_i\cap U_j} = (s_j)_{|U_i\cap U_j}$ (we also say that the family of section $(s_i)_{i\in I}$ satisfies the gluing condition). Then there exists a section $s \in F(U)$ such that $s_i = s_{|U_i}$.*

Remark 1.12 —

1- Let F be a separated presheaf on a topological space. If $F(\emptyset)$ is non-empty than it contains at most one element. Indeed, let s, $t \in F(\emptyset)$. The empty family (i.e., when $I = \emptyset$) is an open covering of the empty open set. As I is empty, the condition $s_{|U_i} = t_{|U_i}$ is true for any $i \in I$. As F is separated, we have $s = t$.

2- If F is a sheaf on a topological space X, $F(\emptyset)$ contains exactly one element. Indeed, by the previous discussion, it suffices to show that $F(\emptyset)$ is not empty. We use again the empty family of opens sets as a covering of the empty open set of X. The empty family of sections satisfies the gluing condition as I is empty. It follows that there exists at least one element s in $F(\emptyset)$ (satisfying to $s_{|U_i} = s_i$ for all $i \in \emptyset$, which is an empty condition).

A morphism of sheaves of sets (groups, rings, modules) $F \to G$ is simply a morphism between the presheaves of sets (groups, rings, modules, etc) F and G.

Lemma 1.13 — *Let X be a topological space and $f : F \to G$ a morphism of sheaves of sets (resp. groups, rings, modules, etc) on X. The following conditions are equivalent:*

(1) *f is invertible.*
(2) *For all $x \in X$, the induced morphism on stalks $f_x : F_x \to G_x$ is invertible.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. We concentrate on the other implication. Assuming (2), we first check injectivity of f on sections. Let s, $t \in F(U)$ such that $f_U(s) = f_U(t)$. As f_x is injective, we must have $s_x = t_x$ for all $x \in U$. Thus, there exists $U_x \in \mathscr{V}(x)$ with $s_{|U_x} = t_{U_x}$. But $(U_x)_{x\in X}$ is a open covering of U. As F is separated, we deduce that $s = t$.

We turn to surjectivity of f on sections. Let $b \in G(U)$. For every $x \in U$, there exists a germ $[a_x, U_x] \in F_x$ (with $a_x \in F(U_x)$) such that $f_x([a_x, U_x]) = b_x$. As $f_x([a_x, U_x])$ is the germ of $f_{U_x}(a_x)$, there exists an open neighborhood $V_x \subset U_x$ of x such that $f_{V_x}((a_x)_{|V_x}) = b_{|V_x}$. In other words, replacing U_x by V_x and a_x by $(a_x)_{|V_x}$, we may assume that $f_{U_x}(a_x) = b_{|U_x}$ for all $x \in U$.

We claim that for x, $y \in U$ we have $(a_x)_{|U_x\cap U_y} = (a_y)_{|U_x\cap U_y}$. As $f_{U_x\cap U_y}$ is injective (by the discussion above), we only need to show that

$$f_{U_x\cap U_y}((a_x)_{|U_x\cap U_y}) = f_{U_x\cap U_y}((a_y)_{|U_x\cap U_y}).$$

But, $f_{U_x\cap U_y}((a_x)_{|U_x\cap U_y}) = (f_{U_x}(a_x))_{|U_x\cap U_y} = b_{U_x\cap U_y}$. Similarly, $f_{U_x\cap U_y}((a_x)_{|U_x\cap U_y}) = b_{U_x\cap U_y}$. This proves our claim.

Now, F is a sheaf. As $(a_x)_{x\in U}$ satisfies the gluing condition, there exists $a \in F(U)$ such that $a_{|U_x} = a_x$ for all $x \in U$. It remains to show that $b = f_U(a)$. Using that G is separated, this follows from $b_{|U_x} = f_{U_x}(a_x) = f_{U_x}(a_{|U_x}) = (f_U(a))_{|U_x}$. This ends the proof of the lemma. □

Remark 1.14 — The first part of the proof of Lemma 1.13 shows that $f : F \to G$ is injective on sections if and only if $f_x : F_x \to G_x$ is injective for all $x \in X$. This property fails for surjectivity. Indeed, there are morphism of sheaves which are not surjective on sections but surjective on stalks.

1.6. **Sheafification of a presheaf.**

Proposition 1.15 — *Let F be a presheaf of sets (resp. groups, rings, modules etc) on a topological space X. There exists a sheaf of sets (resp. groups, rings, modules etc) $\mathrm{a}(F)$ on X together with a morphism $\ell_F : F \to \mathrm{a}(F)$ such that for every morphism $F \to G$ from F to a sheaf G, there exists a unique morphism $\mathrm{a}(F) \to G$ making the triangle*

$$\begin{array}{ccc} F & \longrightarrow & G \\ {\scriptstyle \ell_F}\downarrow & \nearrow & \\ \mathrm{a}(F) & & \end{array}$$

commutative. In other words, the composition with ℓ_F yields a bijection

$$- \circ \ell_F : \hom(\mathrm{a}(F), G) \xrightarrow{\sim} \hom(F, G)$$

for all sheaves of sets (resp. groups, rings, modules, etc) G. Moreover, the couple $(\mathrm{a}(F), \ell_F)$ is unique up to a unique isomorphism.

Proof. We break the proof into steps:
Step 1: The uniqueness of $(\mathrm{a}(F), \ell_F)$ is easy. Indeed, suppose that $(\mathrm{a}'(F), \ell'_F)$ is another couple satisfying to the same condition as in the statement of the proposition. The morphism $\ell'_F : F \to \mathrm{a}'(F)$ can be uniquely factored as

$$\overbrace{F \xrightarrow{\ell_F} \mathrm{a}(F) \overset{u}{\dashrightarrow} \mathrm{a}'(F)}^{\ell'_F} \,.$$

Similarily, the morphism ℓ_F can be uniquely factored as

$$\overbrace{F \xrightarrow{\ell'_F} \mathrm{a}'(F) \overset{u'}{\dashrightarrow} \mathrm{a}(F)}^{\ell_F} \,.$$

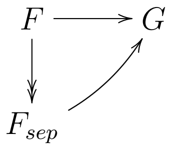
It is clear that $\ell_F = (u' \circ u) \circ \ell_F$ and $\ell'_F = (u \circ u') \circ \ell'_F$. Using the unicity of factorizations, we get that $u' \circ u = \mathrm{id}_{\mathrm{a}(F)}$ and $u \circ u' = \mathrm{id}_{\mathrm{a}'(F)}$. This proves that $\mathrm{a}(F)$ and $\mathrm{a}'(F)$ are canonically isomorphic.
Step 2: Here we construct a separated presheaf F_{sep} out of F and check a universal property. Let $U \subset X$ be an open subset. Given $s, t \in F(U)$, we say that s and t are locally equals and write $s \sim t$ if there exists an open covering $(U_i)_{i \in I}$ of U such that $s_{|U_i} = t_{|U_i}$ for all $i \in I$. This relations is clearly reflexive and symmetric. For transitivity, let $r \in F(U)$ be a third section such that $t \sim r$. Let $(V_j)_{j \in J}$ be a covering of U such that $t_{|V_j} = r_{|V_j}$ for all $j \in J$. Now, the family $(U_i \cap V_j)_{(i,j) \in I \times J}$ is an open covering of U. Moreover, for $(i, j) \in I \times J$ one has

$$s_{|U_i \cap V_j} = t_{|U_i \cap V_j} = r_{|U_i \cap V_j}.$$

This shows that $s \sim t$. Let $F_{sep}(U)$ be the set (resp. group, ring, modules) of equivalence classes of $s \in F(U)$ with respect to this relation. We have restriction maps $F_{sep}(V) \to F_{sep}(U)$ that send the class of $s \in F(V)$ to the class of $s_{|U}$. This makes F_{sep} a presheaf on X et we have an obvious morphism of presheaves $F \to F_{sep}$ by sending a section $s \in F(U)$ to its class.

Let us show that F_{sep} is a separated presheaf. Let $[s]$, $[t] \in F_{sep}(U)$ be the class of sections s, $t \in F(U)$. Suppose that there is a covering $(W_k)_{k\in K}$ of U so that $[s]_{|W_k} = [t]_{|W_k}$ for all $k \in K$. This means that $s_{|W_k} \sim t_{|W_k}$. We can thus find an open covering $(U_{ki})_{i\in I_k}$ of W_k such that $s_{|U_{ki}} = t_{|U_{ki}}$. This means that s and t have the same restrictions to the open subsets U_{ki} which cover U. Thus, we have proven that $[s] = [t]$.

Finally, let G be a separated presheaf and $f : F \to G$ be a morphism of presheaves. If s, $t \in F(U)$ are locally equals, so are $f(s)$, $f(t) \in G(U)$. As G is separated, $f(s) = f(t)$. It follows that $f(s)$ does not depends on the representative of the equivalence class $[s]$. Thus, there is a unique morphism of presheaves $F_{sep} \to G$ making the triangle



commutative. This gives a bijection $\hom(F_{sep}, G) \simeq \hom(F, G)$. Moreover, to prove the proposition, we may assume that F is separated.

Step 3: From now on, we assume our presheaf F to be separated. Here we construct a new presheaf $\mathrm{L}(F)$. The proof that $\mathrm{L}(F)$ is a sheaf will be postponed to another step.

Let U be an open subset of X. A *locally defined section* on U is a couple $(s_i, U_i)_{i\in I}$ where $(U_i)_{i\in I}$ is an open covering of U and $s_i \in F(U_i)$ satisfying the gluing condition, i.e., such that $(s_i)_{|U_i\cap U_j} = (s_j)_{|U_i\cap U_j}$. Two locally defined sections $(s_i, U_i)_{i\in I}$ and $(t_j, V_j)_{j\in J}$ are said to be equivalent (and we write $(s_i, U_i)_{i\in I} \sim (t_j, V_j)_{j\in J}$) if for all $(i, j) \in I \times J$, we have $(s_i)_{U_i\cap V_j} = (t_j)_{|U_i\cap V_j}$.

The above relation is an equivalence relation. Reflexivity and symmetry are clear. Let us check transitivity. Consider three locally defined sections $(s_i, U_i)_{i\in I}$, $(t_j, V_j)_{j\in J}$ and $(r_k, W_k)_{k\in K}$ over U. Assume that $(s_i, U_i)_{i\in I} \sim (t_j, V_j)_{j\in J}$ and $(t_j, V_j)_{j\in J} \sim (r_k, W_k)_{k\in K}$. We will show that $(s_i, U_i)_{i\in I} \sim (r_k, W_k)_{k\in K}$. Fix $(i_0, k_0) \in I \times K$. For all $j \in J$, we have

$$(s_{i_0})_{|U_{i_0}\cap V_j\cap W_{k_0}} = (t_j)_{|U_{i_0}\cap V_j\cap W_{k_0}} = (r_{k_0})_{|U_{i_0}\cap V_j\cap W_{k_0}}.$$

Now remark that $(U_{i_0} \cap V_j \cap W_{k_0})_{j\in J}$ is an open covering of $U_{i_0} \cap W_{k_0}$. As F is separated, we deduce that

$$(s_{i_0})_{|U_{i_0}\cap W_{k_0}} = (r_{k_0})_{|U_{i_0}\cap W_{k_0}}.$$

Given a locally defined section $(s_i, U_i)_{i\in I}$ over U, we denote $[(s_i, U_i)_{i\in I}]$ its the equivalence class. We denote by $\mathrm{L}(F)(U)$ the set of all equivalence classes of locally defined sections on U. Given an inclusion $U \subset V$ of opens in X, we define a restriction map $\mathrm{L}(F)(V) \to \mathrm{L}(F)(U)$ sending $[(t_j, V_j)_{j\in J}] \in \hat{F}(V)$ to $[((t_j)_{|V_j\cap U}, V_j \cap U)_{j\in J}]$. One immediately checks that we have a presheaf $\mathrm{L}(F)$ on X. Moreover, if F is a presheaf of groups (rings, modules, etc) so is $\mathrm{L}(F)$. Indeed, given two locally defined sections $(s_i, U_i)_{i\in I}$ and $(t_j, V_j)_{j\in J}$ over U with values in a presheaf of groups, one defines

$$[(s_i, U_i)_{i\in I}][(t_j, V_j)_{j\in J}] = [((s_i)_{|U_i\cap V_j}.(t_j)_{|U_i\cap V_j}, U_i \cap V_j)_{(i,j)\in I\cap J}].$$

(Check that this is well defined!). Finally, there is a natural morphism of presheaves $\ell_F : F \to \mathrm{L}(F)$ which sends a section $s \in F(U)$ to the class of $[(s, U)]$.

Step 4: We now check that $\mathrm{L}(F)$ is again a separated presheaf. Let $(s_i, U_i)_{i\in I}$ and $(t_j, V_j)_{j\in J}$ be two locally defined sections over an open $U \subset X$. Let $(W_k)_{k\in K}$ be a covering of U such that

$$[(s_i, U_i)_{i\in I}]_{|W_k} = [(t_j, U_j)_{j\in J}]_{|W_k}$$

for all $k \in K$. This means that for all $(i, j, k) \in I \times J \times K$,

$$(s_i)_{|U_i\cap V_j\cap W_k} = (t_j)_{|U_i\cap V_j\cap W_k}.$$

As $(U_i \cap V_j \cap W_k)_{k\in K}$ is an open covering of $(U_i \cap V_j)$ and F is separated, we deduce that

$$(s_i)_{|U_i\cap V_j} = (t_j)_{|U_i\cap V_j}.$$

This means that $[(s_i, U_i)_{i\in I}] = [(t_j, V_j)_{j\in J}]$.

We now prove that $\mathrm{a}(F)$ is a sheaf. Let $(U_i)_{i\in I}$ be a covering of an open $U \subset X$. Suppose that for every $i \in I$ there is given a locally defined section $(s^i_j, V^i_j)_{j\in J_i}$ on U_i so that $([(s^i_j, V^i_j)_{j\in J_i}])_{i\in I}$ satisfies the gluing condition. The family $(s^i_j, V^i_j)_{ij}$ satisfy also the gluing condition and we clearly have $[(s^i_j, V^i_j)_{ij}]_{|U_{i_0}} = [(s^{i_0}_j, V^{i_0}_j)_{j\in J_{i_0}}]$.

Step 5: It remains to show that the sheaf $\mathrm{L}(F)$ satisfies the universal property. Let G be a sheaf and $f : F \to G$ be a morphism of presheaves. Given a locally defined section $(s_i, U_i)_{i\in I}$ over an open U, we get a locally defined section $(f_{U_i}(s_i), U_i)_{i\in I}$. As G is a sheaf, there is a unique section $f[(s_i, U_i)_{i\in I}] \in G(U)$ which restricts to $f_{U_i}(s_i)$ over U_i. This section depends only on the equivalence class of the locally defined section. We get in this way a morphism $\mathrm{L}(F) \to G$ making commutative the triangle

$$\begin{array}{ccc} F & \longrightarrow & G \\ {\scriptstyle \ell_F}\downarrow & \nearrow & \\ \mathrm{L}(F). & & \end{array}$$

The unicity of this factorization is an easy exercise. □

The following lemma shows that sheafification doesn't affects the stalks of a presheaf.

Lemma 1.16 — *Let F be a presheaf on a topological space X. The morphism $\ell_F : F \to \mathrm{a}(F)$ induces isomorphisms on stalks $F_x \xrightarrow{\sim} \mathrm{a}(F)_x$ for all $x \in X$.*

Proof. First we show that $F_x \to \mathrm{a}(F)_x$ is injective. Let (s, U) and (t, V) be two germs on x having the same class in $\mathrm{a}(F)_x$. We may assume that $U = V$ and that the image of s and t are equal in $\mathrm{a}(F)(U)$. This mean that there exists an open cover $(U_i)_{i\in I}$ of U such that $s_{|U_i} = t_{|U_i}$ for all $i \in I$. Choose $i_0 \in I$ such that $x \in U_{i_0}$. Then $(s, U) \sim (s_{|U_{i_0}}, U_{i_0}) = (t_{|U_{i_0}}, U_{i_0}) \sim (t, U)$. This proves injectivity. For surjectivity, we argue in the same manner. Let (s, U) be a germ of $\mathrm{a}(F)$ on x. There exists a cover $(U_i)_{i\in I}$ of U such that $s = [(s_i)_{i\in I}, (U_i)_{i\in I}]$. Pick up $i_0 \in I$ such that $x \in U_{i_0}$. Then, $(s, U) \sim (s_{|U_{i_0}}, U_{i_0})$ which is the image of $(s_i, U_i) \in F_x$. □

Corollary 1.17 — *Let $f : F \to G$ be a morphism of presheaves on a topological space X. The following conditions are equivalent:*

1. *The induced morphism on sheaves $\mathrm{a}(f) : \mathrm{a}(F) \to \mathrm{a}(G)$ is invertible.*
2. *The induced morphisms on stalks $f_x : F_x \to G_x$ is invertible for all $x \in X$.*

Proof. This follows immediately from Lemma 1.13 and Lemma 1.16. $\square$

1.7. **Direct and inverse images of sheaves.**

Lemma 1.18 — *Let $f : Y \to X$ be a continuous map of topological spaces. Let G be a presheaf on Y. If G is a sheaf then $f_*(G)$ is a sheaf on X.*

Proof. We first show that $f_*(G)$ is separated. Let $U \subset X$ be an open subset and $(U_i)_{i\in I}$ be an open covering of U. Then, $(f^{-1}(U_i))_{i\in I}$ is an open covering of $f^{-1}(U)$. Now, suppose that s, $t \in f_*(G)(U) = G(f^{-1}(U))$ have the same restrictions on the U_i for all $i \in I$. This means that $s_{|f^{-1}(U_i)} = t_{|f^{-1}(U_i)}$ for all $i \in I$. As G is separated, we deduce that $s = t$.

To check that $f_*(G)$ is a sheaf pick up a locally defined section of $f_*(G)$ over U. This is equivalent to give a locally defined section of G of the form $(s_i, f^{-1}(U_i))_{i\in I}$ with $(U_i)_{i\in I}$ an open covering of U. As G is a sheaf and $(f^{-1}(U_i))_{i\in I}$ is an open covering of $f^{-1}(U)$, there exists a unique section $s \in G(f^{-1}(U)) = f_*(G)(U)$ which restricts to s_i on U_i. This proves the lemma. $\square$

Let $f : Y \to X$ be a continuous map of topological spaces. If F is a sheaf on X, $f^*(F)$ fails in general to be a sheaf.

Proposition 1.19 — *Let $f : Y \to X$ be a continuous map of topological spaces. Let F be a sheaf on X and G be a sheaf of Y. There is a cononical bijection*

$$\hom(\mathrm{a}f^*(F), G) \simeq \hom(F, f_*(G)).$$

Proof. Indeed, by Proposition 1.10, we have: $\hom(f^*(F), G) \simeq \hom(F, f_*(G))$. As G is a sheaf, we have from Proposition 1.15: $\hom(\mathrm{a}f^*(F), G) \simeq \hom(f^*(F), G)$. This proves the proposition. $\square$

1.8. **Ringed topological spaces.**

By ring, we will mean commutative and unital ring. Modules are always unital (in the sense that multiplication with the unit of the ring is the identity).

Definition 1.20 — *A* ringed space *is a couple $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf of rings $\mathcal{O}_X$ on $|X|$. The sheaf $\mathcal{O}_X$ will be called the* structural sheaf *and $|X|$ the* underlying topological space.

A morphism of ringed spaces $(q, \theta) : Y \to X$ is a continuous map $q : |Y| \to |X|$ between the underlying topological spaces together with a morphism of sheaves of rings $\theta : \mathcal{O}_X \to q_*\mathcal{O}_Y$. Given a morphism of ringed spaces (q, θ) we deduce from Proposition 1.10 a morphism of sheaves of rings $\theta^\# : \mathrm{a}f^*\mathcal{O}_Y \to \mathcal{O}_X$. Moreover, if $y \in Y$, we have a map on stalks $\theta_y : \mathcal{O}_{X,q(y)} \to \mathcal{O}_{Y,y}$ which sends the germ of $f \in \mathcal{O}_X(U)$ to the germ of $\theta(f) \in \mathcal{O}_Y(q^{-1}(U))$ (where U is an open neighborhood of $q(y)$ in X).

Let $X = (|X|, \mathcal{O}_X)$ be a ringed space. (In the sequel we will often abuse notation by saying X where one should say $|X|$. For example, we will say, let $Y \subset X$ be a subset of X, etc) An $\mathcal{O}_X$-module $\mathcal{M}$ is a sheaf of abelian groups on X endowed with a morphism of sheaves $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$ such that for each $U \subset X$, $\mathcal{M}(U)$ is a $\mathcal{O}_X(U)$-module. We isolate a special class of ringed spaces which are by far the most important.

Definition 1.21 — *A* locally ringed space $X = (|X|, \mathcal{O})$ *is a ringed space such that for any* $x \in X$*, the ring* $\mathcal{O}_{X,x}$ *is a local ring.*

Recall that a local ring is a ring which has a unique maximal ideal. If X is a locally ringed space and $x \in X$, we denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ the residue field. Given a function $f \in \mathcal{O}_X(U)$ and $x \in U$, we define the value of f on x to be the class of (f, U) in the residue field $\kappa(x)$ and we denote it by $f(x)$. We say that f admits a zero on $x \in U$ if $f(x) = 0$. This is equivalent to $f_x \in \mathfrak{m}_x$.

Example 1.22 — Let X be a topological manifold endowed with the sheaf $\mathcal{C}^0_X$ of continuous real functions. Let $x \in X$. Let $\mathfrak{m}_x \subset \mathcal{C}^0_{X,x}$ be the ideal of germs of function vanishing on x. This is the only maximal ideal of $\mathcal{C}^0_{X,x}$. Indeed, let f_x be the germ on x of a continuous function f defined on an open neighborhood U of x. Suppose that $f(x) \neq 0$. Then $V_\epsilon = f^{-1}(]f(x) - \epsilon, f(x) + \epsilon[) \cap U$ is an open neighborhood of x and for $\epsilon > 0$ sufficiently small, f is invertible on V_ϵ. Then $f_x.g_x = 1$ in $\mathcal{C}^0_{X,x}$ with $g = (f_{|V_\epsilon})^{-1}$. The evaluation map $f \rightsquigarrow f(x)$ defines an isomorphism

$$\kappa(x) = \mathcal{C}^0_{X,x}/\mathfrak{m}_x \xrightarrow{\sim} \mathbb{R}.$$

Modulo this isomorphism, the class of f in $\kappa(x)$ coïncides with the usual evaluation of f on x.

Lemma 1.23 — *Let* $X = (|X|, \mathcal{O}_X)$ *be locally ringed space,* $U \subset X$ *an open subset and* $f \in \mathcal{O}_X(U)$*. The set* $\mathrm{D}(f) = \{x \in U;\ f(x) \neq 0\}$ *is an open subset of* U*. Moreover, the restriction* $f_{|\mathrm{D}(f)}$ *is invertible.*

Proof. To prove that $\mathrm{D}(f)$ is open we show that each point $x \in \mathrm{D}(f)$ admits an open neighborhood contained in $\mathrm{D}(f)$. But, the condition $f(x) \neq 0$ is equivalent to the condition f_x is invertible in $\mathcal{O}_{X,x}$, which in turn is equivalent to the existence of an open neighborhood V of x in U such that $f_{|V}$ is invertible in $\mathcal{O}_X^\times(V)$. But then, f_y is invertible for all $y \in V$ which implies that $V \subset \mathrm{D}(f)$.

The above argument implies also that $f_{|\mathrm{D}(f)}$ is invertible. Indeed, for any $x \in \mathrm{D}(f)$ there is an open neighborhood $V_x \subset U$ such that $f_{|V_x}$ is invertible. Let $g_x \in \mathcal{O}_X(V_x)$ be the inverse of $f_{|V_x}$. The family $(g_x, V_x)_{x \in U}$ satisfies the gluing condition. As $\mathcal{O}_X$ is a sheaf, there exists $g \in \mathcal{O}(\mathrm{D}(f))$ such that $g_{|V_x} = g_x$. It follows that $(gf)_{|V_x} = 1$. Using that $\mathcal{O}_X$ is separated, we deduce that $fg = 1$. The lemma is proven. □

Definition 1.24 — *Let* $(q, \theta) : Y \to X$ *be a morphism of ringed topological spaces. Suppose that* X *and* Y *are locally ringed spaces. We say that* (q, θ) *is a morphism of locally ringed spaces if* $\mathfrak{m}_{q(y)} = \theta_y^{-1}(\mathfrak{m}_y)$ *for all* $y \in X$ *(where* $\theta_y : \mathcal{O}_{X,q(y)} \to \mathcal{O}_{Y,y}$ *is the induced morphism on stalks).*

Let $(q, \theta) : Y \to X$ be a morphism of locally ringed spaces. For $y \in Y$, we have an extension of residue fields $\kappa(q(y)) \to \kappa(y)$. Let $f \in \mathcal{O}_X(U)$ such that $q(y) \in U$. Then $f(q(x)) = \theta_U(f)(y)$ modulo the identification of $\kappa(q(y))$ with a subfield of $\kappa(y)$. Thus, we may think of $\theta_U(f)$ as being the composition "$f \circ q$".

1.9. Partially defined sheaves.

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{T}' \subset \mathcal{T}$ a subset. We say that $\mathcal{T}'$ is a generating family for the topology of X if $\mathcal{T}'$ is stable by finite intersection and for any open subset $U \subset X$ and any $x \in U$, there exists $U' \in \mathcal{T}'$ such that $U' \subset U$ and

$x \in U'$. One defines the notion of *partial presheaf* on X with respect to $\mathcal{T}'$ in the same way as for presheaves by using opens in $\mathcal{T}'$ rather than all opens of X. More precisely, a partial presheaf F is the family of sets (resp. groups, rings, modules, etc) $F(U)$ for $U \in \mathcal{T}'$ and restriction morphisms ρ_U^V for $U \subset V$ with U, $V \in \mathcal{T}'$ satisfying $\rho_U^U = \mathrm{id}$ and $\rho_U^V \circ \rho_V^W = \rho_U^W$.

A partial presheaf F on a topological space X endowed with a generating family of opens $\mathcal{T}' \subset \mathcal{T}$ is separated (resp. is a sheaf) if it satisfy the same condition as in Definition 1.11 with U and U_i in $\mathcal{T}'$.

Lemma 1.25 — *Let F be a partial sheaf on a topological space X endowed with a generating family of opens $\mathcal{T}' \subset \mathcal{T}$. There exists, up to a unique isomorphism, a sheaf $\tilde{F}$ on X such that F is isomorphic to the restriction of $\tilde{F}$ to the opens in $\mathcal{T}'$.*

Proof. We first construct a sheaf $\tilde{F}$ on X satisfying the required condition. Let $U \subset X$ be an open subset such that there exists at least one $V \in \mathcal{T}'$ such that $U \subset V$. Let $F_1(U)$ be the set of equivalence classes of pair (s, V) with $V \in \mathcal{T}'$ containing U and $s \in F(V)$. Two such pairs (s, V) and (s', V') are equivalent if there exists $V'' \in \mathcal{T}'$ contained in $V \cap V'$, containing U and such that $s_{|V''} = s'_{|V''}$.

If $U \subset X$ is an open which is not contained in any element of $\mathcal{T}'$, we set $F_1(U) = \emptyset$, (resp. $F_1(U) = \{1\}$ if F is a partial sheaf of groups, $F_1(U) = \mathbb{Z}$ if F is a partial sheaf of rings, $F_1(U) = 0$ if F is a partial sheaf of modules).

If $U \subset V$ are opens in X, we may define a restriction morphism $\rho_U^V : F_1(V) \to F_1(U)$ in the following manner. When V is not contained in any element of $\mathcal{T}'$, $F_1(V)$ is a initial object and we have a unique such morphism. If V is contained in an element of $\mathcal{T}'$ then so is U. Let (s, W) be a couple with $s \in F(W)$, $W \in \mathcal{T}'$ containing V, and thus defining an element of $F_1(V)$. As W also contains U, it defines an element in $F_1(U)$ which only depends on the class of (s, W) in $F_1(V)$. This is our restriction map.

We have a constructed a presheaf F_1 on X such that F is canonically isomorphic to the restriction of F_1 to $\mathcal{T}'$. We define $\tilde{F} = \mathrm{a}(F_1)$ to be the associated sheaf. We still need to check that $\tilde{F}(U) \simeq F(U)$ for all $U \in \mathcal{T}'$. For this, we remember the construction of the associated sheaf.

First, we have $F_1(U) \simeq (F_1)_{sep}(U)$. We only need to check injectivity. Let $s, t \in F_1(U)$ be two sections which are identified in $(F_1)_{sep}(U)$. Then there exists an open covering $(U_i)_{i\in I}$ of U such that $s_{|U_i} = t_{|U_i}$. As $\mathcal{T}'$ is a generating family, we may assume that $U_i \in \mathcal{T}'$. This implies that $s = t$ as F is a separated partial presheaf.

So we only need to check that $(F_1)_{sep}(U) \to \mathrm{a}((F_1)_{sep})(U)$ is bijective. As $(F_1)_{sep}$ is separated, we only need to check surjectivity. But, an element of $\mathrm{a}((F_1)_{sep})(U)$ is an equivalence class of locally defined sections. Let $(s_i, U_i)_{i\in I}$ be a locally defined section on U. As before, we may assume that $U_i \in \mathcal{T}'$. As F is a partial sheaf, there exists $s \in F(U) = (F_1)_{sep}(U)$ such that $s_{|U_i} = s_i$.

To finish the proof of the lemma we still need to check uniqueness of $\tilde{F}$. Let G be a sheaf on X and $u : F \xrightarrow{\sim} G_{|\mathcal{T}'}$. There is a morphism of presheaves $f : F_1 \to G$ as follows. Let $U \subset X$ be an open. If U is not contained in any element of $\mathcal{T}'$, there is only one possibility of f_U. If U is contained in some element of $\mathcal{T}'$, then f sends the equivalence class of (s, V) to the $(u_V(a))_{|U}$. As G is a sheaf, f induce a morphism $\tilde{f} : \tilde{F} = \mathrm{a}(F_1) \to G$. To check that this is an isomorphism, it suffices to show that

f induces isomorphism on the stalks. This is obvious as any point of x admits a cofinal systeme of neighborhoods which are elements of $\mathcal{T}'$. □

Remark 1.26 — Let F and G be partial sheaves on a topological space X endowed with a generating family of opens. Given a morphism of partial sheaves $\alpha : F \to G$, there is an induced morphism of sheaves $\tilde{\alpha} : \tilde{F} \to \tilde{G}$. This follows immediately from the construction. Moreover, the natural application

$$\hom(F, G) \to \hom(\tilde{F}, \tilde{G})$$

sending α to $\tilde{\alpha}$ is a bijection. This is an easy exercise.

2. From Commutative Rings to Schemes

Recall that all rings are supposed commutative and unital and all modules are supposed unital. The zero and the unit of a ring are denoted by 0 and 1 respectively. The zero of a module is also denoted by 0. If the equality $0 = 1$ holds in a ring, the latter consists of one element and is called the zero ring (which we usually denote by 0). Up to isomorphisms, there is only one module over the zero ring, namely the zero module, which is also denoted by 0.

2.1. Modules of fractions.

Let A be a ring and $S \subset A$ a subset of A. We say that S is *multiplicative* if

- $1 \in S$,
- for all $f,\, g \in S$ we have $fg \in S$.

The most important examples of multiplicative sets are $< f >= \{f^n,\, n \in \mathbb{N}\}$ and $A \backslash \mathfrak{p}$ where $\mathfrak{p} \subset A$ is a prime ideal.

Let $S \subset A$ be a multiplicative subset of a ring A. Given an A-module M we may construct a new A-module $S^{-1}M$ in the following way. Define a relation on the set $S \times M$ by setting $(f, x) \sim (g, y)$ if there exists $h \in S$ such that $h(gx - fy) = 0$. Let us check that this is an equivalence relation. The relation is clearly reflexive and symmetric. It is also transitive. Indeed, assume that $(f_1, x_1) \sim (f_2, x_2)$ and $(f_2, x_2) \sim (f_3, x_3)$. Fix h and h' in S such that $h(f_2x_1 - f_1x_2) = h'(f_3x_2 - f_2x_3) = 0$. We then get

$$hh'f_2(f_3x_1 - f_1x_3) = h'f_3h(f_2x_1 - f_1x_2) + hf_1h'(f_3x_2 - f_2x_3) = 0.$$

As a set, $S^{-1}M = S \times M/\sim$ is the set of equivalence classes for the above relation. The equivalence class of a couple (f, x) will be denoted by a fraction $\dfrac{x}{f}$ (and sometimes $f^{-1}x$). We define the addition on $S^{-1}M$ by the formula

$$\frac{x}{f} + \frac{y}{g} = \frac{gx + fy}{fg}.$$

One immediately checks that this is well-defined (i.e., depends only on the equivalence classes). The action of $a \in A$ on $S^{-1}M$ is given by $a\left(\dfrac{x}{f}\right) = \dfrac{ax}{f}$. Moreover, one has a morphism of A-modules $\rho^S : M \to S^{-1}M$ sending $x \in M$ to the fraction $\dfrac{x}{1}$. The couple $(S^{-1}M, \rho^S)$ satisfies a universal property:

Lemma 2.1 — *For $f \in S$, the multiplication by f on $S^{-1}M$ is invertible. Conversely, let N be an A-module such that the multiplication by f on N is invertible for all $f \in S$. For every morphism of A-modules $\alpha : M \to N$ there exists a unique morphism $\tilde{a} : S^{-1}M \to N$ making commutative the triangle*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle \rho^S}\downarrow & \nearrow{\scriptstyle \tilde{\alpha}} & \\ S^{-1}M. & & \end{array}$$

In other words, the application $-\circ \rho^S : \hom(S^{-1}M, N) \to \hom(M,N)$ *is bijective.*

Proof. The multiplication by f on $S^{-1}M$ admits an inverse which sends a fraction $\frac{x}{g} \in S^{-1}M$ to $\frac{x}{fg}$. For the other statement, denote $i_{N,f} : N \to N$ the inverse of the multiplication by f on N. We define $\tilde{\alpha}(\frac{x}{g}) = i_{N,g}(\alpha(x))$. This is clearly the unique morphism of A-modules such that $\tilde{\alpha} \circ \rho^S = \alpha$. □

Given a multiplicative set $T \subset A$ containing S, we have a unique morphism of A-modules $\rho_S^T : S^{-1}M \to T^{-1}M$ making commutative the triangle

$$\begin{array}{ccc} M & \xrightarrow{\rho^T} & T^{-1}M \\ {\scriptstyle \rho^S}\downarrow & \nearrow {\scriptstyle \rho_S^T} & \\ S^{-1}M. & & \end{array}$$

This follows immediately from Lemma 2.1. The morphism ρ_S^T sends a fraction $\frac{x}{f}$ to the same fraction considered as an element of $T^{-1}M$.

Lemma 2.2 — *Let* $S \subset T \subset A$ *be multiplicative subsets. Assume that for every* $g \in T$ *there exists* $h \in A$ *such that* $gh \in S$*. Then for any* A*-module* M*, the canonical morphism* $\rho_S^T : S^{-1}M \to T^{-1}M$ *is invertible.*

Proof. Remark that ρ_S^T is surjective. Indeed, given $\frac{y}{g} \in T^{-1}M$ and $h \in A$ such that $gh \in S$, we have $\frac{y}{g} = \frac{hy}{hg}$ and the latter is in the image of ρ_S^T. On the other hand, $\rho_S^T(\frac{x}{f}) = 0$ implies that there exists $g \in T$ such that $gx = 0$. If $h \in A$ is such $gh \in S$, we get also $(hg)x = 0$. This proves that $\frac{x}{f} = 0$ in $S^{-1}M$. □

Given a mutiplicative subset S there is maximal multiplicative subset T containing S and satisfying the condition of Lemma 2.2, namely the set $\hat{S}$ of divisors of elements in f:

$$\hat{S} = \{g \in A;\ \exists h \in A,\ gh \in S\}.$$

A multiplicative subset S is called *saturated* if $S = \hat{S}$. Of course, $\hat{S}$ is itself a saturated multiplicative subset.

Given a morphism of A-modules $\alpha : M \to N$, there is a unique morphism of A-modules $S^{-1}\alpha : S^{-1}M \to S^{-1}N$ such that

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle \rho^S}\downarrow & & \downarrow{\scriptstyle \rho^S} \\ S^{-1}M & \xrightarrow{S^{-1}\alpha} & S^{-1}N \end{array}$$

commutes. If α is surjective, so is $S^{-1}\alpha$. If α is injective, so is $S^{-1}\alpha$ as it follows from the following lemma:

Lemma 2.3 — *Let M be an A-module. The kernel of $\rho^S : M \to S^{-1}M$ consists of the elements $x \in M$ such that $0 \in S.x = \{fx,\, f \in S\}$. Moreover, the natural morphism*

$$S^{-1}M \to S^{-1}(M/\ker(\rho^S)) \tag{1}$$

is an isomorphism.

Proof. The kernel of ρ^S consists of $x \in M$ such that $\frac{x}{1} = 0$. This means that $(1, x) \sim (1, 0)$ which is equivalent to the existence of $f \in S$ with $0 = f(1.x - 1.0) = fx$.

Let us show that (1) is invertible. Surjectivity is clear. For injectivity, we consider a fraction $\frac{x}{f}$ in the kernel of (1). As multiplication by f is invertible in $S^{-1}(M/\ker(\rho^S))$, we know also that $\frac{x}{1}$ is in the kernel of (1). By the first part of the lemma, there exists $g \in S$ such that $gx \in \ker(\rho^S)$. Thus, there is $h \in S$ such that $hgx = 0$. This implies that $\frac{x}{f} = 0$. □

Notation 2.4 — If $f \in A$, we denote $< f >^{-1} M$ by M_f. If $\mathfrak{p} \subset A$ is a prime ideal, we denote $(A \backslash \mathfrak{p})^{-1}M$ by $M_{\mathfrak{p}}$.

2.2. Rings of fractions.

Let A be a ring and $S \subset A$ a multiplicative subset. As A is an A-module in an obvious way, we may consider the A-module $S^{-1}A$. It is easily checked that $S^{-1}A$ is a ring where the multiplication is given by

$$\left(\frac{a}{f}\right)\left(\frac{b}{g}\right) = \frac{ab}{fg}.$$

The ring $S^{-1}A$ is called the ring of S-fractions in A. Moreover, $\rho^S : A \to S^{-1}A$ is a morphism of rings. Also, given a multiplicative subset $T \subset A$ containing S, the natural morphism $\rho^T_S : S^{-1}A \to T^{-1}A$ is a morphism of rings.

Given an A-module M, $S^{-1}M$ is in a natural way an $S^{-1}A$-module for the action given by

$$\left(\frac{a}{f}\right)\left(\frac{x}{g}\right) = \frac{ax}{fg}$$

where $a \in A$, $x \in M$ and $f,\, g \in S$.

Lemma 2.5 — *Let $\mathfrak{a} \subset S^{-1}A$ be an ideal. Then, $\mathfrak{a} = S^{-1}\mathfrak{a}_0$ with $\mathfrak{a}_0 = (\rho^S)^{-1}(\mathfrak{a})$.*

Proof. We know that $S^{-1}\mathfrak{a}_0 \to S^{-1}A$ is injective and its image is contained in $\mathfrak{a}$. To show that $S^{-1}\mathfrak{a}_0 \simeq \mathfrak{a}$, take $\frac{a}{f} \in \mathfrak{a}$. Then, $\frac{a}{1} = f.\left(\frac{a}{f}\right) \in \mathfrak{a}$. It follows that $a \in \mathfrak{a}_0$ so that $\frac{a}{f} \in S^{-1}\mathfrak{a}_0$. This proves the lemma. □

Recall that an ideal $\mathfrak{p} \subset A$ is a prime ideal if $1 \notin \mathfrak{p}$ and for every $f,\, g \in A$ we have the implication: $fg \in \mathfrak{p} \Rightarrow f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. This is equivalent to ask that $A \backslash \mathfrak{p}$ is a multiplicative set.

Proposition 2.6 — *Let $\mathfrak{p} \subset A$ be a prime ideal. Exactly one of the following alternatives holds:*

(1) $S \cap \mathfrak{p} = \emptyset$, *in which case,* $S^{-1}\mathfrak{p}$ *is a prime ideal.*
(2) $S \cap \mathfrak{p} \neq \emptyset$, *in which case,* $S^{-1}\mathfrak{p} = S^{-1}A$.

Moreover, every prime ideal $\mathfrak{q}$ *of* $S^{-1}A$ *is equal to* $S^{-1}\mathfrak{p}$ *where* $\mathfrak{p} = (\rho^S)^{-1}(\mathfrak{q})$.

Proof. If $S \cap \mathfrak{p}$ contains an element f, then $1 = \frac{f}{f} \in S^{-1}\mathfrak{p}$ which implies that $S^{-1}\mathfrak{p} = S^{-1}A$. We may assume that $S \cap \mathfrak{p} = \emptyset$. It follows that $1 \notin S^{-1}\mathfrak{p}$. To show that $S^{-1}\mathfrak{p}$ is prime, choose $\frac{a}{f}, \frac{b}{g} \in S^{-1}A$ such that $\frac{ab}{fg} \in S^{-1}\mathfrak{p}$. Thus, we may write

$$\frac{ab}{fg} = \frac{c}{h}$$

with $c \in \mathfrak{p}$ and $h \in S$. This means that there exists $k \in S$ such that $khab = kfgc \in \mathfrak{p}$. As $kh \notin \mathfrak{p}$ we have that $ab \in \mathfrak{p}$ and we may assume that $a \in \mathfrak{p}$. This implies that $\frac{a}{f} \in S^{-1}\mathfrak{p}$. The last statement follows from Lemma 2.5. □

Let $\theta : A \to B$ be a morphism of rings. Then $\theta(S) \subset B$ is a multiplicative subset of B and we have a canonical morphism of rings $\tilde{\theta} : S^{-1} \to \theta(S)^{-1}B$ which sends a fraction $\frac{a}{f}$ to $\frac{\theta(a)}{\theta(f)}$

Corollary 2.7 — *Let* S *be a multiplicative subset. Then* $\hat{S} = (\rho^S)^{-1}((S^{-1}A)^\times)$ *is the inverse image of the set of invertible elements in* $S^{-1}(A)$. *We also have* $\hat{S} = A\setminus \cup_{\mathfrak{p}\cap S=\emptyset}\, \mathfrak{p}$.

Proof. Indeed, let $g \in A$ so that $\frac{g}{1}$ is invertible in $S^{-1}A$. Then there exists $\frac{a}{f}$ such that $\frac{ag}{f} = \frac{1}{1}$. Thus, we may find $h \in S$ such that $hag = hf$. This proves that $g \in \hat{S}$.

Now, $(S^{-1}A)^\times$ is the complement of all the prime ideals. It follows that

$$\hat{S} = (\rho^S)^{-1}\left(S^{-1}A\setminus \bigcup_{\mathfrak{p}\cap S=\emptyset} S^{-1}\mathfrak{p}\right) = A\setminus \bigcup_{\mathfrak{p}\cap S=\emptyset} \mathfrak{p}.$$

This proves the corollary. □

Remark 2.8 — Given two multiplicative subsets $S, T \subset A$ of a ring A, we denote $ST = \{fg;\ f \in S \text{ and } g \in T\}$. This is a multiplicative subset of A. Put also $T' = \rho^S(T)$. There is a canonical isomorphism $T'^{-1}(S^{-1}A) \simeq (ST)^{-1}A$ sending a fraction $\frac{f^{-1}a}{1^{-1}g}$ to $\frac{a}{fg}$ for $a \in A$, $f \in S$ and $g \in T$.

2.3. **The spectrum of a ring.**

Let A be a ring. We denote by $\mathsf{Spec}(A)$ the set of prime ideals of A. Given a subset $E \subset A$, we define $\mathcal{Z}(E)$ to be the subset of $\mathsf{Spec}(A)$ consisting of primes ideals containing E. This is sometimes called the *zero set* of E. If (E) denotes the ideal of A generated by the elements of E, we have $\mathcal{Z}(E) = \mathcal{Z}((E))$.

Lemma 2.9 —

0- $\mathcal{Z}(A) = \emptyset$ *and* $\mathcal{Z}(0) = \mathsf{Spec}(A)$. *If* $E \subset E' \subset A$, *then* $\mathcal{Z}(E') \subset \mathcal{Z}(E)$.

1- *If* $\mathfrak{a}, \mathfrak{b} \subset A$ *are ideals, we have* $\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) = \mathcal{Z}(\mathfrak{a}\mathfrak{b}) = \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b})$.

2- *If* $(\mathfrak{a}_i)_{i \in I}$ *is a family of ideals in* A*, we have* $\mathcal{Z}(\sum_{i \in I} \mathfrak{a}_i) = \cap_{i \in I} \mathcal{Z}(\mathfrak{a}_i)$.

Proof. The first and last assertions are obvious.

We clearly have $\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) \subset \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) \subset \mathcal{Z}(\mathfrak{a}\mathfrak{b})$. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$. Assume that $\mathfrak{a} \not\subset \mathfrak{p}$. Then, there exists $a \in \mathfrak{a}$ not contained in $\mathfrak{p}$. Now, if $b \in \mathfrak{b}$, we get $ab \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$. As $\mathfrak{p}$ is prime, we necessarily have $b \in \mathfrak{p}$. Thus, we have proven that $\mathfrak{b} \subset \mathfrak{p}$, i.e., $\mathfrak{p} \in \mathcal{Z}(\mathfrak{b})$. Similarly, if $\mathfrak{b} \not\subset \mathfrak{p}$ we get that $\mathfrak{p} \in \mathcal{Z}(\mathfrak{a})$. This means that $\mathcal{Z}(\mathfrak{a}\mathfrak{b}) \subset \mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b})$. □

Definition 2.10 — *A subset of the form* $\mathcal{Z}(\mathfrak{a})$ *is called a* Zariski closed *subset of* $\mathsf{Spec}(A)$*. A subset of* $\mathsf{Spec}(A)$ *is called* Zariski open *if it is the complement of a Zariski closed subset. By Lemma 2.9, the set of Zariski opens is a topology on* $\mathsf{Spec}(A)$ *called the* Zariski topology.

Given a subset $Z \subset \mathsf{Spec}(A)$, we set $\mathcal{I}(Z) = \cap_{\mathfrak{p} \in Z} \mathfrak{p}$. (When Z is the empty set, this intersection is taken to be A.) If T is another subset of $\mathsf{Spec}(A)$, we have $\mathcal{I}(Z \cup T) = \mathcal{I}(Z) \cap \mathcal{I}(T)$.

Recall that the radical $\sqrt{\mathfrak{a}}$ of ideal $\mathfrak{a} \subset A$ is the set of $a \in A$ such that $a^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. It is obvious that $\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\sqrt{\mathfrak{a}})$. Moreover, we have:

Proposition 2.11 — *Let* $\mathfrak{a} \subset A$ *be an ideal. We have* $\sqrt{\mathfrak{a}} = \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$.

Proof. We need to show that $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}$ (where $\mathfrak{p}$ are prime ideals). Replacing A by $A/\mathfrak{a}$, we may assume that $\mathfrak{a} = (0)$. In other words, it suffices to show that the nil-radical $\mathcal{N}il(A)$ (i.e., the set of nilpotent elements) of A is the intersection of the prime ideals of A. We clearly have $\mathcal{N}il(A) \subset \cap_{\mathfrak{p} \in \mathsf{Spec}(A)} \mathfrak{p}$. To show the inverse inclusion, let $a \in A$ be contained in all the prime ideals of A and let's show that a is nilpotent. Consider the ring of fractions A_a. By Proposition 2.6, A_a has no prime ideals. This happens only when $A_a = 0$ is the zero ring. Using Lemma 2.3 and the fact that 1 is in the kernel of $A \to A_a$, we deduce that $0 \in \{a^n, n \in \mathbb{N}\}$. This means that a is nilpotent. □

Corollary 2.12 — *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals in* A*. Then,* $\mathcal{Z}(\mathfrak{a}) \subset \mathcal{Z}(\mathfrak{b})$ *if and only if* $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$.

Corollary 2.13 — *There is a bijection from the set of ideals equal to their radicals and the set of Zariski closed subsets of* $\mathsf{Spec}(A)$*. This bijection is strictly decreasing for the inclusion of ideals and subsets.*

For $f \in A$, we denote by $\mathrm{D}(f) \subset \mathsf{Spec}(A)$ the complement of $\mathcal{Z}(f)$. This is a Zariski open subset consisting of the prime ideals $\mathfrak{p}$ such that $f \notin \mathfrak{p}$. The family $(\mathrm{D}(f))_{f \in A}$ is a generating family for the Zariski topology on $\mathsf{Spec}(A)$ (see subsection 1.9 for the meaning of *generating family*). Indeed, we clearly have $\mathrm{D}(f) \cap \mathrm{D}(g) = \mathrm{D}(fg)$ and given a Zariski open set U equal to the complement of $\mathcal{Z}(\mathfrak{a})$, U is the union of $\mathrm{D}(f)$ for $f \in \mathfrak{a}$. Moreover, by Proposition 2.6, there is a canonical bijection

$$l_f : \mathrm{D}(f) \xrightarrow{\sim} \mathsf{Spec}(A_f) \qquad (2)$$

sending a prime ideal $\mathfrak{p}$ not containing f to the prime ideal $\mathfrak{p}_f$ of the ring of fractions A_f. This bijection is a homeomorphism for the Zariski topologies. This follows easily from Lemma 2.5 and the fact that $l_f(\mathcal{Z}(\mathfrak{a}_0) \cap \mathrm{D}_f) = \mathcal{Z}(\mathfrak{a})$ (where $\mathfrak{a}$ is an ideal of A_f and $\mathfrak{a}_0 = (\rho^{<f>})^{-1}(\mathfrak{a})$).

Lemma 2.14 — *Let f and $(f_i)_{i\in I}$ be elements of A. Then $\mathrm{D}(f) \subset \bigcup_{i\in I} \mathrm{D}(f_i)$ if and only if there exists a family $(g_i)_{\in I}$ of elements in A which are zero except for a finite number of indices and such that $\sum_{i\in I} f_i g_i = f^r$ for some $r \in \mathbb{N}$. In particular, there is a finite subset $I_0 \subset I$ such that $\mathrm{D}(f) \subset \bigcup_{i\in I_0} \mathrm{D}(f_i)$.*

Proof. The condition is sufficient. Indeed, if $\mathfrak{p}$ is a prime ideal not containing f, then $f_i \notin \mathfrak{p}$ for at least one $i \in I$. Let us prove that the necessity of the condition.

Let $\mathfrak{a} = (f_i, i \in I)$ be the ideal generated by the f_i. The condition $\mathrm{D}(f) \subset \bigcup_{i\in I} \mathrm{D}(f_i)$ is equivalent to $\mathcal{Z}(\mathfrak{a}) \subset \mathcal{Z}(f)$ which, by Corollary 2.12, is equivalent to $f \in \sqrt{\mathfrak{a}}$. This means that $f^r \in \mathfrak{a}$ for some $r \in \mathbb{N}$. □

Remark 2.15 — Let X be a topological space. We say that X is quasi-compact if for every open covering $(U_i)_{i\in I}$ of U there exists a finite subset $I_0 \subset I$ such that $U = \cup_{i\in I_0} U_i$. The last statement of Lemma 2.14 can be rephrased by saying that the opens $\mathrm{D}(f)$ are quasi-compact.

For an element $f \in A$, se denote by $\mathrm{S}(f)$ the saturated multiplicative subset associated to $< f >$. This is the set of divisors of powers of f.

Corollary 2.16 — *Let f and g be elements of A. Then $\mathrm{D}(f) \subset \mathrm{D}(g)$ if and only if $\mathrm{S}(g) \subset \mathrm{S}(f)$.*

2.4. The affine scheme associated to a ring.

We keep the notation from the previous section. Let A be a ring. We want to construct a sheaf of rings on the topological space $\mathsf{Spec}(A)$. More generally, we will associate to an A-module M a sheaf of A-modules $\widetilde{M}$ on $\mathsf{Spec}(A)$.

The first step is to construct a partial presheaf for the generating family $(\mathrm{D}(f))_{f\in A}$. Recall that for $f \in A$ we denote by $\mathrm{S}(f)$ the set of all divisors of powers of f. This is the saturated multiplicative subset associated to $< f >= \{f^n; n \in \mathbb{N}\}$. We also have $\mathrm{S}(f) = A \setminus \cup_{f\notin\mathfrak{p}} \mathfrak{p} = A \setminus \cup_{\mathfrak{p}\in \mathrm{D}(f)} \mathfrak{p}$. The obvious map $M_f \to \mathrm{S}(f)^{-1}M$ is an isomorphism of A-modules.

By Corollary 2.16, we have $\mathrm{D}(f) \subset \mathrm{D}(g)$ if and only if $\mathrm{S}(g) \subset \mathrm{S}(f)$. We define a partial presheaf $\widetilde{M}^{part}$ (with respect to the generating family $(\mathrm{D}(f))_{f\in A}$) by sending an open $\mathrm{D}(f)$ to the A-module of fractions $\mathrm{S}(f)^{-1}M$ and an inclusion $\mathrm{D}(f) \subset \mathrm{D}(g)$ to the obvious morphism $\rho^{\mathrm{S}(f)}_{\mathrm{S}(g)} : \mathrm{S}(g)^{-1}M \to \mathrm{S}(f)^{-1}M$. It is clear that $\widetilde{A}^{part}$ is also a partial presheaf of rings. It will be denoted by $\mathcal{O}^{part}$. It is also clear that $\widetilde{M}^{part}$ is in a natural way an $\mathcal{O}^{part}$-module. We have the following important fact.

Proposition 2.17 — *$\widetilde{M}^{part}$ is a partial sheaf.*

Proof. We first check that $\widetilde{M}^{part}$ is a separated partial presheaf. Let $\dfrac{s}{f^n} \in \widetilde{M}(\mathrm{D}(f)) = \mathrm{S}(f)^{-1}M$. Let $(\mathrm{D}(f_i))_{i\in I}$ be an open covering of $\mathrm{D}(f)$ such that $\left(\dfrac{s}{f^n}\right)_{|\mathrm{D}(f_i)} = 0$ for all $i \in I$. By the last statement of Lemma 2.14, we may assume that I is finite. For all $i \in I$, there is $n_i \in \mathbb{N}$ such that $f_i^{n_i} s = 0$. Using Lemma 2.14, we may find

$g_i \in A$ such that $f^r = \sum_{i\in I} g_i f_i^{n_i}$. It follows that $f^r s = 0$ which in turn implies that $\dfrac{s}{f^n} = 0$.

Let's now check the gluing property for $\widetilde{M}^{part}$. Let $(\mathrm{D}(f_i))_{i\in I}$ be an open covering of $\mathrm{D}(f)$. Suppose that for each $i \in I$ we are given a section $\dfrac{s_i}{f_i^{n_i}}$ and that the gluing condition is satisfied. This means that for all $(i,j) \in I^2$, there is $m_{ij} \in \mathbb{N}$ such that $(f_i f_j)^{m_{ij}}(f_i^{n_i} s_j - f_j^{n_j} s_i) = 0$. Using the last statement of Lemma 2.14 we will again assume that I is finite.

Let $m \in \mathbb{N}$ be an integer bigger than all the m_{ij}. By Lemma 2.14, we may find $g_i \in A$ such that $f^r = \sum_{i\in I} g_i f_i^{n_i+m}$. Let $s = \sum_{i\in I} g_i f_i^m s_i$ and consider the section $\dfrac{s}{f^r} \in \widetilde{M}^{part}(\mathrm{D}(f)) = \mathrm{S}(f)^{-1}M$. We will prove that $\dfrac{s}{f_r}$ restricts to $\dfrac{s_j}{f_j^{n_j}}$ on $\mathrm{D}(f_j)$, i.e.,

$$\frac{s}{f^r} = \frac{s_j}{f_j^{n_j}} \qquad \text{in} \quad \mathrm{S}(f_j)^{-1}M,$$

for all $j \in I$. For this, we compute

$$f_j^{n_j} s - f^r s_j = f_j^{n_j} \sum_{i\in I} g_i f_i^m s_i - \sum_{i\in I} g_i f_i^{n_i+m} s_j = \sum_{i\in I} g_i f_i^m (f_j^{n_j} s_i - f_i^{n_i} s_j).$$

It follows that $f_j^m(f_j^{n_j} s - f^r s_j) = 0$. This proves the proposition. □

By Lemma 1.25, the partial sheaf $\widetilde{M}^{part}$ can be uniquely extended into a sheaf of A-modules $\widetilde{M}$ on $\mathsf{Spec}(A)$. Similarly, $\mathcal{O}^{part}$ can be uniquely extended into a sheaf of rings $\mathcal{O}$ on $\mathsf{Spec}(A)$. Given an open subset $U \subset \mathsf{Spec}(A)$ and an open covering $(\mathrm{D}(f_i))_{i\in I}$ of U, we may write a section of $\widetilde{M}$ as an equivalence class of locally defined section, i.e., of families $\left(\dfrac{s_i}{f_i^{n_i}}\right)_{i\in I} \in \prod_{i\in I} \mathrm{S}(f_i)^{-1}M$, satisfying the gluing condition. A similar statement holds also for $\mathcal{O}$. In particular, one sees that $\widetilde{M}$ is an $\mathcal{O}$-module in a canonical way. Moreover, given a morphism of A-modules $\alpha : M \to N$ there is an induced morphism $\widetilde{\alpha} : \widetilde{M} \to \widetilde{N}$ of $\mathcal{O}$-module sending a section $\dfrac{s}{f^n} \in \widetilde{M}(\mathrm{D}(f))$ to the section $\dfrac{\alpha(s)}{f^n} \in \widetilde{N}(\mathrm{D}(f))$.

Definition 2.18 — *The ringed space* $(\mathsf{Spec}(A), \mathcal{O})$ *is called the* affine scheme *associated to A. We often abuse notation and denote by* $\mathsf{Spec}(A)$ *this ringed space.*

Remark 2.19 — If $f \in A$, we have a canonical isomorphism $\mathcal{O}(\mathrm{D}(f)) \simeq A_f$. In particular, $\mathcal{O}(\mathsf{Spec}(A)) \simeq A$. More generally, if M is an A-module, we have a canonical isomorphism $\widetilde{M}(\mathrm{D}(f)) \simeq M_f$.

Lemma 2.20 — *Let $\mathfrak{p} \subset A$ be a prime ideal. The stalk $\mathcal{O}_{\mathfrak{p}}$ is canonically identified with the ring $A_{\mathfrak{p}}$. More generally, if M is an A-module, $\widetilde{M}_{\mathfrak{p}} \simeq M_{\mathfrak{p}}$.*

Proof. The stalk $\mathcal{O}_{\mathfrak{p}}$ is the ring of germs of sections $\dfrac{a}{f^n} \in \mathrm{S}(f)^{-1}A$ for $\mathfrak{p} \in \mathrm{D}(f)$. As $\mathrm{D}(f)$ is the set of prime ideals such that $f \notin \mathfrak{p}$, we may define a morphism $\mathcal{O}_{\mathfrak{p}} \to A_{\mathfrak{p}}$

by sending a germ $\left(\frac{a}{f^n}\right)_{\mathfrak{p}}$ to the fraction $\frac{a}{f^n}$ considered as an element of $A_{\mathfrak{p}}$. It is an easy exercise to check that this is an isomorphism of rings. The same proof works for A-modules. $\square$

Corollary 2.21 — *$(\mathsf{Spec}(A), \mathcal{O})$ is a locally ringed space. Moreover, via the identification $\mathcal{O}_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$, the maximal ideal $\mathfrak{m}_{\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{p}}$ corresponds to $\mathfrak{p}A_{\mathfrak{p}}$.*

Remark 2.22 — Let $U \subset \mathsf{Spec}(A)$ be an open subset and $f \in \mathcal{O}(U)$. Recall that for $\mathfrak{p} \in U$, the value of f at $\mathfrak{p}$, denoted by $f(\mathfrak{p})$, is the class of f in the residue field $\kappa(\mathfrak{p}) = \mathcal{O}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$. In Lemma 1.23, we defined $\mathrm{D}(f)$ to be the set of prime ideals $\mathfrak{p}$ such that $f(\mathfrak{p}) \neq 0$. When $U = \mathsf{Spec}(A)$, we get the same $\mathrm{D}(f)$ defined as the complement of $\mathcal{Z}(f)$.

Lemma 2.23 — *Let $U \subset \mathsf{Spec}(A)$ be a quasi-compact Zariski open subset and $n \in \mathcal{O}(U)$. Then n is nilpotent if and only if $n(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in U$.*

Proof. As U is quasi-compact, it may be covered by a finite number of opens of the form $\mathrm{D}(f)$. Using that $\mathcal{O}$ is a sheaf, we may assume that $U = \mathrm{D}(f)$. Then $n(\mathfrak{p}) = 0$ implies that n is an element of all the prime ideals of A_f. By Proposition 2.11, n is nilpotent. $\square$

Recall that a morphism of sheaves on a topological space is injective if it is injective on sections. It is called surjective, if it is surjective on stalks. A morphism of sheaves which is injective and surjective is invertible.

Lemma 2.24 — *Let A be a ring and $\alpha : M \to N$ a morphism of A-modules. Then α is injective (resp. surjective, invertible) if and only if $\tilde{\alpha} : \widetilde{M} \to \widetilde{N}$ is an injective (resp. surjective, invertible) morphism of $\mathcal{O}$-modules.*

Proof. The statement for injectivity is obvious. We only treat surjectivity. If $M \to N$ is surjective, so is $M_{\mathfrak{p}} \to M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset A$. Now assume that $\widetilde{M} \to \widetilde{N}$ is surjective. Let K be the cokernel of $M \to N$. We have exact sequences

$$M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}} \longrightarrow K_{\mathfrak{p}} \longrightarrow 0$$

which show that $K_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subset A$. As $\widetilde{K}_{\mathfrak{p}} = K_{\mathfrak{p}}$, we deduce that the sheaf $\widetilde{K}$ is zero. But $K \simeq \widetilde{K}(\mathsf{Spec}(A))$. The proof of the lemma is complete. $\square$

Proposition 2.25 — *Let $X = (|X|, \mathcal{O}_X)$ be a locally ringed space. To give a morphism of locally ringed spaces $X \to \mathsf{Spec}(A)$ is equivalent to give a morphism of rings $A \to \mathcal{O}_X(|X|)$. In other words, there is a natural bijection*

$$\hom(X, \mathsf{Spec}(A)) \simeq \hom(A, \mathcal{O}_X(|X|)).$$

Proof. Let $(q, \theta) : X \to \mathsf{Spec}(A)$ be a morphism of locally ringed spaces. The morphism $\theta : \mathcal{O} \to q_*\mathcal{O}_X$ induces a morphism $A \simeq \mathcal{O}(\mathsf{Spec}(A)) \to \mathcal{O}_X(q^{-1}\mathsf{Spec}(A)) = \mathcal{O}_X(|X|)$. This gives the natural application

$$\hom(X, \mathsf{Spec}(A)) \to \hom(A, \mathcal{O}_X(|X|)). \tag{3}$$

Let us construct the inverse of this application. Assume we are given a morphism of rings $\alpha : A \to \mathcal{O}_X(|X|)$. Let x be a point of X. The inverse image of the maximal ideal $\mathfrak{m}_x$ by the composition $A \to \mathcal{O}_X(|X|) \to \mathcal{O}_{X,x}$ is a prime ideal of A which we denote by $q_\alpha(x)$. This gives an application $q_\alpha : |X| \to \mathsf{Spec}(A)$.

To prove that q_α is continuous, it is sufficient to check that for $f \in A$, the set $q_\alpha^{-1}(\mathrm{D}(f))$ is open. By construction, $q_\alpha^{-1}(\mathrm{D}(f))$ is the set of $x \in X$ such that f is not in the ideal $q_\alpha(x)$, i.e., $\alpha(f)(x) \neq 0$. In other words, $q_\alpha^{-1}(\mathrm{D}(f)) = \mathrm{D}(\alpha(f))$. The latter is open by Lemma 1.23.

Now, we need to define a morphism of sheaves of rings $\mathcal{O} \to (q_\alpha)_*\mathcal{O}_X$. By Remark 1.26, it is sufficient to define a morphism on the associated partial presheaves, i.e., a family of morphisms $\mathcal{O}(\mathrm{D}_f) \to \mathcal{O}_X(q_\alpha^{-1}(\mathrm{D}_f))$ compatible with the restriction maps. Using Lemma 2.1, and the fact that $\alpha(f)_{|q_\alpha^{-1}(\mathrm{D}(f))}$ is invertible, we see that there is a unique morphism $\theta^\alpha_{\mathrm{D}(f)}$ making commutative the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & \mathcal{O}_X(|X|) & \longrightarrow & \mathcal{O}(q_\alpha^{-1}(\mathrm{D}(f))). \\ \downarrow & & & \nearrow & \\ A_f & & & & \end{array}$$

Compatibility with the restriction maps is easily checked. We denote by $\theta^\alpha : \mathcal{O} \to (q_\alpha)_*\mathcal{O}_X$ the morphism just obtained. It is clear that the couple $(q_\alpha, \theta^\alpha) : X \to \mathsf{Spec}(A)$ is a morphism of locally ringed spaces. This gives an application

$$\hom(A, \mathcal{O}_X(|X|)) \to \hom(X, \mathsf{Spec}(A)) \tag{4}$$

which is clearly a section to (3).

To end the proof of the proposition, we may show that (3) is injective, i.e., that a morphism $(q, \theta) : X \to \mathsf{Spec}(A)$ of locally ringed spaces is completely determined by the induced morphism of rings $A \to \mathcal{O}_X(|X|)$. First, this is the case for the map q. Indeed, as $\theta_x : A_{q(x)} = \mathcal{O}_{q(x)} \to \mathcal{O}_{X,x}$ is a local morphism of local rings, we should have $\theta_x^{-1}(\mathfrak{m}_x) = q(x)A_{q(x)}$. This implies that the $q(x)$ is the inverse image of $\mathfrak{m}_x$ by $A \to A_{q(x)} \to \mathcal{O}_{X,x}$ which is also $A \to \mathcal{O}_X(|X|) \to \mathcal{O}_{X,x}$. Also, θ is completely determined by the condition that the squares

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{O}(\mathrm{D}(f)) \simeq A_f \\ \downarrow & & \downarrow \\ \mathcal{O}_X(|X|) & \longrightarrow & \mathcal{O}_X(q^{-1}(\mathrm{D}(f))) \end{array}$$

commute, as follows from the uniqueness in Lemma 2.1. □

Corollary 2.26 — *Let A and B be two rings. Given a morphism of rings $\alpha : A \to B$, there is an induced morphism of locally ringed spaces $(q_\alpha, \theta^\alpha) : \mathsf{Spec}(B) \to \mathsf{Spec}(A)$ such that:*

- *q_α sends a prime ideal $\mathfrak{p} \subset A$ to the prime ideal $\alpha^{-1}(\mathfrak{p})$,*
- *for $f \in A$, we have $q_\alpha^{-1}(\mathrm{D}(f)) = \mathrm{D}(\alpha(f))$. Moreover, the morphism $\theta^\alpha_{|\mathrm{D}(f)} : \mathcal{O}(\mathrm{D}(f)) \to \mathcal{O}(\mathrm{D}(\alpha(f)))$ corresponds, via the identifications $\mathcal{O}(\mathrm{D}(f)) \simeq A_f$ and $\mathcal{O}(\mathrm{D}(\alpha(f))) \simeq B_{\alpha(f)}$, to the canonical morphism $\alpha_f : A_f \to B_{\alpha(f)}$.*

Moreover, the association $\alpha \rightsquigarrow (q_\alpha, \theta^\alpha)$ gives a bijection

$$\hom(A, B) \xrightarrow{\sim} \hom(\mathsf{Spec}(B), \mathsf{Spec}(A))$$

from the set of morphisms of rings from A to B to the set of morphisms of locally ringed spaces from $\mathsf{Spec}(B)$ *to* $\mathsf{Spec}(A)$.

2.5. General schemes and quasi-coherent modules.

Let $X = (|X|, \mathcal{O}_X)$ be a ringed space. If $|U|$ is an open set of $|X|$, we write $\mathcal{O}_U$ for the restriction of $\mathcal{O}_X$ to the open subsets included in $|U|$. Then $(|U|, \mathcal{O}_U)$ is again a ringed space. If X is locally ringed, then so is $(|U|, \mathcal{O}_U)$.

Definition 2.27 — *A ringed space $X = (|X|, \mathcal{O}_X)$ is called an* affine scheme *if it is isomorphic to the spectrum of a ring. It is called a* scheme *if every point $x \in X$ admits an open neighborhood $|U|$ such that $(|U|, \mathcal{O}_U)$ is an affine scheme. An open subset $|U| \subset |X|$ such that $(|U|, \mathcal{O}_U)$ is an affine scheme is called an* affine open subset.

By definition, schemes are locally ringed spaces. Morphisms of schemes are simply morphisms of locally ringed spaces. This is the natural definition in view of Proposition 2.25.

Remark 2.28 — Let $X = (|X|, \mathcal{O}_X)$ be a locally ringed space. Then X is an affine scheme if and only if the natural morphism $X \to \mathsf{Spec}(\mathcal{O}_X(|X|))$ (from Proposition 2.25) is an isomorphism of ringed spaces. Also X is a scheme if and only if any $x \in X$ admits an open neighborhood $|U|$ such that the canonical morphism $(|U|, \mathcal{O}_U) \to \mathsf{Spec}(\mathcal{O}_X(|U|))$ is an isomorphism of ringed spaces.

Notation 2.29 — From now on, a scheme will be simply denoted by capital letters X, Y, Z, etc. The topological space underlying a scheme X will be denoted by $|X|$ or simply X if no confusion can arise. The structural sheaf of a scheme X will be denoted by $\mathcal{O}_X$. Morphisms of schemes will be denoted by $q : Y \to X$ and $(q, \theta) : Y \to X$ if we need to be precise.

Remark 2.30 — Let U be an affine scheme and $\mathcal{M}$ an $\mathcal{O}_U$-module. Set $A = \mathcal{O}_U(U)$ and $M = \mathcal{M}(U)$ which is an A-module. Denote by $\iota : U \to \mathsf{Spec}(A)$ the natural morphism which is invertible (as U is affine). There is an obvious morphism $\iota^*\widetilde{M} \to \mathcal{M}$ as follows. If $f \in \mathcal{O}_U(U)$, the A-module $i^*\widetilde{M}(\mathrm{D}(f))$ is canonically isomorphic to M_f. As multiplication by f is invertible on $\mathcal{M}(\mathrm{D}(f))$, there exists by Lemma 2.1 a unique morphism $\iota^*\widetilde{M}(\mathrm{D}(f)) \simeq M_f \to \mathcal{M}(\mathrm{D}(f))$ making commutative the square

$$\begin{array}{ccc} M & = & \mathcal{M}(U) \\ \downarrow & & \downarrow \\ \iota^*\widetilde{M}(\mathrm{D}(f)) & \longrightarrow & \mathcal{M}(\mathrm{D}(f)). \end{array}$$

It is easily checked that the morphisms $\iota^*\widetilde{M}(\mathrm{D}(f)) \to \mathcal{M}(\mathrm{D}(f))$ commute with restriction maps. By Remark 1.26, we get in this way a morphism of $\mathcal{O}_U$-modules $\iota^*\widetilde{M} \to \mathcal{M}$. Furthermore, this morphism is invertible if and only if for all $f \in A$, the natural morphism $\mathcal{M}(U)_f \to \mathcal{M}(\mathrm{D}(f))$ is invertible.

Definition 2.31 — *Let X be a scheme and $\mathcal{M}$ and $\mathcal{O}_X$-module. We say that $\mathcal{M}$ is* quasi-coherent *if any $x \in X$ admits an affine open neighborhood U such that one of the following three equivalent conditions is satisfied:*

(1) $\mathcal{M}_{|U}$ *is isomorphic to the pull-back via* $U \xrightarrow{\sim} \mathsf{Spec}(\mathcal{O}_X(U))$ *of the* $\mathcal{O}$*-module* $\widetilde{N}$ *associated to an* $\mathcal{O}_X(U)$*-module* N*.*
(2) $\mathcal{M}_{|U}$ *is isomorphic to the pull-back via* $U \xrightarrow{\sim} \mathsf{Spec}(\mathcal{O}_X(U))$ *of the* $\mathcal{O}$*-module* $\widetilde{\mathcal{M}(U)}$*.*
(3) *For all* $f \in \mathcal{O}_X(U)$*, the obvious morphism* $\mathcal{M}(U)_f \to \mathcal{M}(\mathrm{D}(f))$ *is invertible.*

The equivalence between the three conditions of Definition 2.31 follows easily from Remark 2.30. We have the following important result.

Proposition 2.32 — *Let* X *be an affine scheme and* $\mathcal{M}$ *a quasi-coherent* $\mathcal{O}_X$*-module. Then* $\mathcal{M}$ *is canonically isomorphic to (the pull-back via* $X \simeq \mathsf{Spec}(\mathcal{O}_X(X))$ *of)* $\widetilde{\mathcal{M}(X)}$*. Moreover, given two quasi-coherent* $\mathcal{O}_X$*-modules* $\mathcal{M}$ *and* $\mathcal{N}$*, there is a canonical bijection*

$$\hom(\mathcal{M}, \mathcal{N}) \simeq \hom(\mathcal{M}(X), \mathcal{N}(X)).$$

Proof. We may assume that $X = \mathsf{Spec}(A)$. We denote by M the A-module $\mathcal{M}(X)$. There is an obvious morphism of $\mathcal{O}$-modules $\widetilde{M} \to \mathcal{M}$ defined as in Remark 2.30. We need to check that $\widetilde{M} \to \mathcal{M}$ is an isomorphism. We may do this locally (by Lemma 1.13).

By hypothesis, we know that there exists an open covering $\mathrm{D}(f_i)_{i\in I}$ of $\mathsf{Spec}(A)$ such that $\mathcal{M}_{|\mathrm{D}(f_i)} \simeq \widetilde{\mathcal{M}(\mathrm{D}(f_i))}$. We may assume that I is finite and we argue by induction on its cardinal. If I contains one element, there is nothing to prove.

Suppose that $I = \{1, 2\}$. As $\mathcal{M}$ is a sheaf, we have an exact sequence of A-modules

$$0 \longrightarrow M \xrightarrow{(1)} \mathcal{M}(\mathrm{D}(f_1)) \oplus \mathcal{M}(\mathrm{D}(f_2)) \xrightarrow{(2)} \mathcal{M}(\mathrm{D}(f_1 f_2))$$

where the first map sends a gobal section s to the vector $(s_{|\mathrm{D}(f_1)}, s_{|\mathrm{D}(f_2)})$ and the second map sends a vector (s, t) to $s_{|\mathrm{D}(f_1 f_2)} - t_{|\mathrm{D}(f_1 f_2)}$. Taking $< f_1 >$-fractions, we get an exact sequence

$$0 \longrightarrow M_{f_1} \longrightarrow \mathcal{M}(\mathrm{D}(f_1)) \oplus \mathcal{M}(\mathrm{D}(f_2))_{f_1} \longrightarrow \mathcal{M}(\mathrm{D}(f_1 f_2)) .$$

As $\mathcal{M}_{|\mathrm{D}(f_2)} \simeq \widetilde{\mathcal{M}(\mathrm{D}(f_2))}$, we see that $\mathcal{M}(\mathrm{D}(f_2))_{f_1} \to \mathcal{M}(\mathrm{D}(f_1 f_2))$ is invertible. It follows that $M_{f_1} \to \mathcal{M}(\mathrm{D}(f_1))$ is invertible. By symmetry, we also have that $M_{f_2} \to \mathcal{M}(\mathrm{D}(f_2))$ is invertible. This shows that $M \to \mathcal{M}$ is invertible.

Now assume that I contains more than 2 elements. Fix $i_0 \in I$ and let $I' = I \backslash \{i_0\}$. As $(\mathrm{D}(f_i))_{i\in I}$ is a covering of $\mathsf{Spec}(A)$ we may find a family $(g_i)_{i\in I}$ such that $\sum_i g_i f_i = 1$. Replacing f_i by $f_i g_i$, we may assume that $\sum_{i\in I} f_i = 1$. Let $g = \sum_{i\neq i_0} f_i$. Then, we have an open covering $\mathsf{Spec}(A) = \mathrm{D}(f_{i_0}) \cup \mathrm{D}(g)$. Moreover, $\mathrm{D}(f_i)_{i\neq i_0}$ is an open covering of $\mathrm{D}(g)$. Thus, by induction we know that $\mathcal{M}_{|\mathrm{D}(g)} \simeq \widetilde{\mathcal{M}(\mathrm{D}(g))}$. Thus we are reduced to the case where I contains 2 elements and which we already treated.

The fact that $\hom(\mathcal{M}, \mathcal{N}) \to \hom(\mathcal{M}(X), \mathcal{N}(X))$ is bijective, is left as an easy exercise. □

Corollary 2.33 — *Let* X *be a scheme and* $\mathcal{M}$ *an* $\mathcal{O}_X$*-module. Then* $\mathcal{M}$ *is quasi-coherent if for any affine open subset* $U \subset X$ *and any* $f \in \mathcal{O}_X(U)$*, the obvious morphism* $\mathcal{M}(U)_f \to \mathcal{M}(\mathrm{D}(f))$ *is invertible.*

Example 2.34 — Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ a subsheaf of ideals. We say that $\mathcal{I}$ is a quasi-coherent ideal if the $\mathcal{O}_X$-module $\mathcal{I}$ is quasi-coherent. If $\mathcal{I}$ is quasi-coherent, so is the sheaf of ideals $\sqrt{\mathcal{I}}$ given by $\sqrt{\mathcal{I}}(U) = \sqrt{\mathcal{I}(U)}$ for $U \subset X$ quasi-compact. Indeed, we may assume that $X = \mathsf{Spec}(A)$ is affine and that $\mathcal{I} = \tilde{I}$ for $I \subset A$ an ideal. Then for $f \in A$, we need to check that $(\sqrt{I})_f = \sqrt{I_f}$ which is an easy exercise. When $\sqrt{0} = 0$, we say that X is a *reduced* scheme. By Lemma 2.23, X is reduced if and only if for all opens $U \subset X$ and $f \in \mathcal{O}_X(U)$, the condition $f(x) = 0$ for all $x \in U$ implies that $f = 0$.

2.6. **Open and closed immersions.**

Definition 2.35 — *Let $j : Y \to X$ be a morphism of schemes. We say that j is an open immersion if:*

(1) $|j| : |Y| \to |X|$ *is isomorphic to the inclusion of an open subset of* $|X|$,
(2) *for all* $y \in Y$, $\mathcal{O}_{X,j(y)} \to \mathcal{O}_{Y,y}$ *is an isomorphism.*

Lemma 2.36 — *Let X be a scheme and $U \subset X$ an open subset. Then U is a scheme when endowed with restriction of the structural sheaf of X. Moreover, $U \to X$ is an open immersion and any open immersion is isomorphic to such a one.*

Proof. That $(U, (\mathcal{O}_X)_{|U}) \to (X, \mathcal{O}_X)$ is an open immersion is clear. Let $j : Y \to X$ be a general open immersion. Set $U = j(Y)$. This is an open subset of X. We clearly have a factorization of j as a composition

$$(Y, \mathcal{O}_Y) \xrightarrow{u} (U, (\mathcal{O}_X)_{|U}) \to (X, \mathcal{O}_X).$$

To see that u is an isomorphism, we need to check that $u^*(\mathcal{O}_X)_{|U} \to \mathcal{O}_Y$ is an invertible morphism of sheaves on Y. For this, it is sufficient to look at the induced morphism on stalks. But, for $y \in Y$ this morphism is simply $\mathcal{O}_{X,j(y)} \to \mathcal{O}_{Y,y}$ which is invertible by hypothesis. □

Definition 2.37 — *Let X be a scheme. We say that X is* quasi-affine *if there exists an open immersion $X \to X'$ with X' an affine scheme.*

Except when the contrary is explicitly mentioned, an open subset U of a scheme X will always be considered as a scheme endowed with the restriction of the structural sheaf of X. We generally abuse notation by saying that U is a open subscheme of X. Given a morphism of schemes $f : Y \to X$, there is a unique morphism of schemes $f_U : f^{-1}(U) \to U$ making the square

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f_U} & U \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

commutative (where the vertical arrows are the obvious open immersions).

Definition 2.38 — *Let $j : Y \to X$ be a morphism of schemes. We say that j is a closed immersion if any point $x \in X$ admits an affine open neighborhood U such that $j^{-1}(U)$ is affine and the morphism $\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ is surjective.*

Lemma 2.39 — *Let $j : Y \to X$ be a closed immersion. Then $j(|Y|)$ is a closed subset of $|X|$ and the topology of $|Y|$ is the one induced from the topology of $|X|$.*

Proof. The property to check is local on X. Thus, we may assume that X is affine, Y is affine and $\mathcal{O}_X(X) = A \to \mathcal{O}_Y(Y) = B$ surjective. Then j can be identified with $\mathsf{Spec}(A/\mathfrak{a}) \to \mathsf{Spec}(A)$ where $\mathfrak{a} = \ker(A \to B)$. Therefor, the image of j is simply the set $\mathcal{Z}(\mathfrak{a})$ of prime ideals $\mathfrak{p} \subset A$ containing $\mathfrak{a}$. □

Lemma 2.40 — *Let $j : Y \to X$ be a morphism of schemes. Then j is a closed immersion if and only if for every affine open subscheme $U \subset X$, $j^{-1}(U)$ is an affine open sub-scheme of Y and the natural morphism $\mathcal{O}_X(U) \to \mathcal{O}_Y(j^{-1}(U))$ is surjective.*

Proof. The condition is clearly sufficient. Let's proof that it is necessary. For this, we may assume that $X = \mathsf{Spec}(A)$ is affine. Consider the sheaf $j_*\mathcal{O}_Y$. We have a natural morphism $\theta : \mathcal{O}_X \to j_*\mathcal{O}_Y$ which makes $j_*\mathcal{O}_Y$ an $\mathcal{O}_X$-module. We claim that $j_*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-module.

Indeed, as j is a closed immersion, there exists an open covering $(U_i)_{i\in I}$ of X such that $j^{-1}(U_i)$ is affine. It follows that for any $f \in \mathcal{O}_X(U_i)$, we have $j_*(\mathcal{O}_Y)(\mathrm{D}(f)) = \mathcal{O}_Y(j^{-1}(\mathrm{D}(f))) = \mathcal{O}_Y(\mathrm{D}(\theta_{U_i}(f))) \simeq \mathcal{O}(j^{-1}(U_i))_f = (j_*\mathcal{O}_Y)(U_i)_f$. This proves our claim.

Using Proposition 2.32, we have $j_*\mathcal{O}_Y = \widetilde{B}$ where $B = \mathcal{O}_Y(Y)$. The natural morphism $A \to B$ is surjective as $\mathcal{O}_X \to j_*\mathcal{O}_Y$ is. To finish the proof we still need to show that Y is affine, i.e., the natural morphism $Y \to \mathsf{Spec}(B)$ is an isomorphism. We may check this locally. The verification is then an easy exercise. □

Proposition 2.41 —

1- *Let $i : Z \to X$ be a closed immersion. We define a subseaf of ideals $\mathcal{I}_Z \subset \mathcal{O}_X$ by setting $\mathcal{I}_Z(U) = \ker(\mathcal{O}_X(U) \to \mathcal{O}_Z(i^{-1}(U))$. Then $\mathcal{I}_Z(U)$ is a quasi-coherent sheaf of ideals.*

2- *Conversely, let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There exists, up to a unique isomorphism, a closed immersion $i : Z \to X$ such that $\mathcal{I}_Z = \mathcal{I}$. Moreover, $i(|Z|)$ is the set of points $x \in X$ such that $\mathcal{I}_x \subset \mathfrak{m}_x$.*

Proof. Let's prove the first part. It is clear that $\mathcal{I}_Z$ is a sheaf of ideals in $\mathcal{O}_X$. To check that $\mathcal{I}_Z$ is quasi-coherent, we may assume that X is affine. Then i is isomorphic to $\mathsf{Spec}(A) \to \mathsf{Spec}(A/\mathfrak{a})$ and $\mathcal{I}_Z$ is the sheaf $\tilde{\mathfrak{a}}$.

We pass to the second part. We define $|Z| \subset |X|$ to be the set of $x \in X$ such that $\mathcal{I}_x \subset \mathfrak{m}_x$. This is a closed subset of $|X|$. We endow $|Z|$ with the induced topology.

Consider the presheaf of rings $\mathcal{O}_X/\mathcal{I}$ and let $\mathcal{K} = \mathrm{a}(\mathcal{O}_X/\mathcal{I})$ be the associated sheaf. It is clear that $\mathcal{K}$ is a quasi-coherent $\mathcal{O}_X$-module and also a sheaf of rings endowed with a morphism $\mathcal{O}_X \to \mathcal{K}$. If $U \subset X$ is open and affine, we have $\mathcal{K}(U) = \mathcal{O}_X(U)/\mathcal{I}(U)$. In particular, when $U \cap |Z| = \emptyset$, $\mathcal{K}(U) = 0$.

Denote by $i : |Z| \to |X|$ the obvious inclusion. We define the structure sheaf $\mathcal{O}_Z$ as $i^*\mathcal{K}$. If $U \subset X$ is open, the canonical morphism

$$\mathcal{K}(U) \to \mathcal{O}_Z(U \cap |Z|) = \mathcal{K}(U \cap |Z|)$$

is an isomorphism. Indeed, we have an inverse $\mathcal{K}(U \cap Z) \to \mathcal{K}(U)$ which sends a germ $[s, V]$ to the global section associated to the locally defined section $(0 \in F(U\backslash|Z|), s_{|V\cap U})$ on U. In other words, $\mathcal{K} \simeq i_*\mathcal{O}_Z$ which implies also that $\mathcal{O}_Z$ is a sheaf.

We have a morphism of locally ringed spaces $(i, \theta) : (|Z|, \mathcal{O}_Z) \to (|X|, \mathcal{O}_X)$ where θ is the composition $\mathcal{O}_X \to \mathcal{K} \simeq i_*\mathcal{O}_Z$. It is an easy exercise to check that this is indeed a closed immersion with the expected properties. $\square$

Example 2.42 — Let X be a scheme and $|Z| \subset |X|$ a closed subset. There is a sheaf of ideals $\mathcal{I}_{|Z|} \subset \mathcal{O}_X$ such that for any affine open subscheme $U \subset X$, $\mathcal{I}_{|Z|}(U)$ is the intersection of all the prime ideals in $\mathcal{O}_X(U)$ which are in the image of $|Z| \cap U$ by the canonical isomorphism $U \to \mathsf{Spec}(\mathcal{O}_X(U))$. One easily checks that $\mathcal{I}_{|Z|}$ is a quasi-coherent sheaf of ideals and that the associated closed immersion $Z \to X$ has $|Z|$ as underlying closed subset. Moreover, the subscheme Z is reduced, and is the unique (up to a unique isomorphism) reduced subscheme having $|Z|$ as underlying topological space.

Recall that a subset $Y \subset X$ of a topological space X is called locally closed if for every $y \in Y$, there exists an open neighborhood U of y in X such that $Y \cap U$ is a closed subset of U. This is equivalent to ask that Y is a closed subset of some open subset of X.

Definition 2.43 — *Let $j : Y \to X$ be a morphism of schemes. We say that j is a locally closed immersion if*

(1) *$|j| : |Y| \to |X|$ is isomorphic to the inclusion of a locally closed subset of $|X|$ and the topology on $|Y|$ is the one induced by $|j|$,*
(2) *for all $y \in Y$, the natural morphism $\mathcal{O}_{X,j(y)} \to \mathcal{O}_{Y,y}$ is surjective.*

Lemma 2.44 — *Let $j : Y \to X$ be a morphism of schemes. Then j is a locally closed immersion if an only if for every open subscheme $U \subset X$ such that $j(|Y|) \cap |U|$ is closed in $|U|$, the morphism $j^{-1}(U) \to U$ is a closed immersion.*

Proof. The property we need to prove is local on X. Thus, we may assume that $X = \mathsf{Spec}(A)$ is an affine scheme. Replacing Y by $j^{-1}(U)$, we may assume that $j(|Y|) \subset |U|$. Furthermore, if $(Y_i)_{i \in I}$ is an open covering of Y, it is sufficient to prove the claim for the locally closed immersions $Y_i \to X$. Indeed, let $U_i \subset U$ be opens such that $Y_i \subset U_i$ are closed. Then, using the covering $(U_i)_{i \in I} \bigcup (U - Y)$ of U, we see that $Y \to U$ is a closed immersion.

Thus, we may assume that $Y = \mathsf{Spec}(B)$ and $X = \mathsf{Spec}(A)$ are affine schemes and that $Y \to X$ is given by a morphism of rings $\alpha : A \to B$. We know that for all prime ideals $\mathfrak{q} \subset B$, $A_{\alpha^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is surjective.

For every $\mathfrak{q} \subset B$, we may find $f \notin \alpha^{-1}(\mathfrak{q})$ such that $j(|Y|) \cap \mathrm{D}(f)$ is closed in $\mathrm{D}(f)$. As $j^{-1}(\mathrm{D}(f)) = \mathrm{D}(\alpha(f))$, the latter is an affine open sub-scheme of Y. Thus replacing $X = \mathsf{Spec}(A)$ by $\mathrm{D}(f) \simeq \mathsf{Spec}(A_f)$ and Y by $\mathrm{D}(\alpha(f)) \simeq \mathsf{Spec}(B_{\alpha(f)})$ we may assume that $j(|Y|) \subset |X|$ is a closed subset.

We will show that under these conditions, $A \to B$ is surjective (which clearly finish the proof of the lemma). It is sufficient to show that for all prime ideal $\mathfrak{p} \subset A$ we have that $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is surjective. When $\mathfrak{p} = \alpha^{-1}(\mathfrak{q})$ this is our assumption. Suppose that $\mathfrak{p}$ is not the inverse image of any prime ideal of B. This means that $\mathfrak{p}$ is not in the closed subset $|Y|$ of $|X|$. Thus, we may find $f \in A$ such that $f \notin \mathfrak{p}$ and $\mathrm{D}(\alpha(f)) = j^{-1}(\mathrm{D}(f)) = \emptyset$. This means that the image of f in B is nilpotent. But then, $B_{\mathfrak{p}} = 0$ and the surjectivity is clear in this case. $\square$

Corollary 2.45 — *Let $j : Y \to X$ be a morphism of schemes. The following conditions are equivalent:*

(1) *j is a closed immersion,*
(2) *j is a locally closed immersion and $j(|Y|)$ is closed in $|X|$.*

Lemma 2.46 — *Let $i : Z \to Y$ and $j : Y \to X$ be two morphisms of schemes.*
1- *If i and j are closed (resp. locally closed) immersions then so does $j \circ i$.*
2- *If $j \circ i$ is a locally closed immersion then so does i.*

Proof. The first part is an easy exercise. For the second part, the question is local on Z. Thus, we may assume that Z, Y and X are affine and that $j \circ i$ is a closed immersion. The claim is then obvious. □

Remark 2.47 — The analogous statement of the second part of Lemma 2.46 for closed immersions is false in general.

2.7. Gluing schemes and fiber products.

Lemma 2.48 — *Let $(X_i)_{i \in I}$ be a family of schemes. Assume that for every couple $(i, j) \in I^2$ we are given an open subscheme $X_{ij} \subset X_i$ and an isomorphism $f_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$ such that:*

(1) *$X_{ii} = X_i$ and $f_{ii} = \mathrm{id}_{X_{ii}}$.*
(2) *For i, j, $k \in I$ we have $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$ and the diagram*

$$\begin{array}{ccc} X_{ij} \cap X_{ik} & \xrightarrow{f_{ij}} & X_{ji} \cap X_{jk} \\ & \searrow^{f_{ik}} & \downarrow f_{jk} \\ & & X_{kj} \cap X_{ki} \end{array}$$

commutes.

Then, there exists (up to a unique isomorphism) a scheme X together with a family of open immersions $u_i : X_i \to X$ such that the diagrams

$$\begin{array}{ccccc} X_{ij} & \xrightarrow[\sim]{f_{ij}} & X_{ji} & \longrightarrow & X_j \\ \downarrow & & & & \downarrow u_j \\ X_i & & \xrightarrow{u_i} & & X \end{array}$$

commute and the equalities $X = \cup_{i \in I} u_i(X_i)$ and $u_i(X_{ij}) = u_i(X_i) \cap u_j(X_j)$ hold.

Proof. We first construct a topological space $|X|$. Consider the topological space $Y = \coprod_{i \in I} |X_i|$. A point of Y is a couple (x_i, i) where $i \in I$ and $x_i \in |X_i|$. We define on Y an equivalence relation by setting $(x_i, i) \sim (x_j, j)$ if $x_i \in X_{ij}$, $x_j \in X_{ji}$ and $f_{ij}(x_i) = x_j$. This is an equivalence relation. Transitivity is obvious. Symmetry follows from the fact that $f_{ij} \circ f_{ji} = f_{ii} = \mathrm{id}$. For transitivity, we pick a third couple (x_k, k) such that $(x_j, j) \sim (x_k, k)$. Using that $f_{ij}(x_i) = x_j$ and $f_{jk}(x_j) = x_k$, we conclude that $f_{ik}(x_i) = f_{jk} \circ f_{ij}(x_i) = f_{jk}(x_j) = x_k$.

Define $|X| = Y/\sim$ to be the quotient set endowed with the quotient topology. Then, the obvious maps $u_i : |X_i| \to |X|$ are open immersions and the diagram

$$\begin{array}{ccccc} |X_{ij}| & \xrightarrow[\sim]{|f_{ij}|} & |X_{ji}| & \longrightarrow & |X_j| \\ \downarrow & & & & \downarrow{\scriptstyle u_j} \\ |X_i| & & \xrightarrow{\quad u_i \quad} & & |X| \end{array}$$

commute. Moreover, we clearly have $u_i(|X_{ij}|) = u_i(|X_i|) \cap u_j(|X_j|)$.

It remains to construct the structural sheaf on $|X|$. Let $|U| \subset |X|$ be an open subset. We define $\mathcal{O}_X(|U|)$ to be the set of families $(s_i \in \mathcal{O}_{X_i}(u_i^{-1}(|U|)))_{i \in I}$ such that $(s_i)_{|X_{ij} \cap u_i^{-1}(|U|)} = (s_j)_{|X_{ji} \cap u_j^{-1}(|U|)} \circ f_{ij}$. It is easily seen that this gives a sheaf of rings on X and that the restriction of $\mathcal{O}_X$ to $u_i(X_i)$ is canonically isomorphic to $\mathcal{O}_{X_i}$ (modulo the identification $X_i \simeq u_i(X_i)$). It follows that $(X, \mathcal{O}_X)$ is a scheme. The uniqueness of $(X, \mathcal{O}_X)$ is easy. It can be derived from the universal property described in the next remark. □

Remark 2.49 — The scheme $(X, \mathcal{O}_X)$ constructed in Lemma 2.48 is called the *scheme obtained by gluing the X_i along the open subschemes X_{ij}*. It has the following universal property. Let Y be any scheme. To give a morphism form $h : X \to Y$, is equivalent to give a family of morphism $h_i : X_i \to Y$ such that the diagram

$$\begin{array}{ccccc} X_{ij} & \xrightarrow{f_{ij}} & X_{ji} & \longrightarrow & X_j \\ \downarrow & & & & \downarrow{\scriptstyle h_j} \\ X_i & & \xrightarrow{\quad h_i \quad} & & Y \end{array}$$

commutes.

We will use Lemma 2.48 to construct fiber products of schemes. Given a diagram of sets

$$\begin{array}{ccc} & & B_1 \\ & & \downarrow{\scriptstyle f_1} \\ B_2 & \xrightarrow{f_2} & A, \end{array}$$

the fiber product $(B_1 \times_A B_2)$ is the subset of $B_1 \times B_2$ consisting of the couples (b_1, b_2) such that $f_1(b_1) = f_2(b_2)$. Let C be another set. To give a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g_1} & B_1 \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_2 & \longrightarrow & A \end{array}$$

is equivalent to give a morphism $h : C \to B_1 \times_A B_2$. We pass from one data to another by $h(c) = (g_1(c), g_2(c))$ and $g_i(c) = pr_i(h(c))$ where pr_i is the projection to the i-th factor.

We seek for a similar formalism for schemes. We make a definition:

Definition 2.50 — *Suppose we are given two morphisms of schemes $f_i : Y_i \to X$ for $i \in \{1, 2\}$. A triple (Z, p_1, p_2) consisting of a scheme Z and morphisms $p_i : Z \to Y_i$ is called a* fiber product *if the following conditions are satisfied.*

(1) *The square*

$$\begin{array}{ccc} Z & \xrightarrow{p_1} & Y_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y_2 & \xrightarrow{f_2} & X \end{array}$$

commutes.

(2) *For any scheme T, the canonical application*

$$\hom(T, Z) \to \hom(T, Y_1) \times_{\hom(T,X)} \hom(T, Y_2)$$

is bijective.

For practical reasons, we also introduce the following notion:

Definition 2.51 — *A square of morphisms of schemes*

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is called cartesian *if it makes T a fiber product of Y and Z over X. In other words, the square is commutative and for every scheme U the canonical morphism*

$$\hom(U, T) \to \hom(U, Y) \times_{\hom(U,X)} \hom(U, Z)$$

is bijective.

If a fiber product of schemes exists than it is unique (up to a unique isomorphism) as it follows from the universal property. We denote it by $Y_1 \times_X Y_2$.

Lemma 2.52 — *Assume that $X = \mathsf{Spec}(A)$, $Y_1 = \mathsf{Spec}(B_1)$ and $Y_2 = \mathsf{Spec}(B_2)$ are affine schemes. Then the fiber product $Y_1 \times_X Y_2$ exists and is given by*

$$Y_1 \times_X Y_2 = \mathsf{Spec}(B_1 \otimes_A B_2).$$

Moreover, the projections $Y_1 \times_X Y_2 \to Y_i$ are induced by the obvious morphisms $B_i \to B_1 \otimes_A B_2$.

Proof. Using Proposition 2.25, we see that to give a commutative diagram of schemes (or more generally, of locally ringed spaces)

$$\begin{array}{ccc} T & \longrightarrow & \mathsf{Spec}(B_1) \\ \downarrow & & \downarrow \\ \mathsf{Spec}(B_2) & \longrightarrow & \mathsf{Spec}(A) \end{array}$$

is equivalent to give a commutative diagram of rings

$$\begin{array}{ccc} \mathcal{O}_T(T) & \longleftarrow & B_1 \\ \uparrow & & \uparrow \\ B_2 & \longleftarrow & A \end{array}$$

Using the universal property for tensor products of rings, this is equivalent to give a morphism of rings $B_1 \otimes_A B_2 \to \mathcal{O}_T(T)$. We get the lemma by applying a second time Proposition 2.25. $\square$

Proposition 2.53 — *Fiber products of schemes exist.*

Proof. Let $f : Y \to X$ and $g : Z \to X$ be two morphism of schemes. We shall construction $Y \times_X Z$ in four steps.

Step 1: Let $X_0 \subset X$, $Y_0 \subset Y$ and $Z_0 \subset Z$ be open subschemes such that $f(Y_0) \subset X_0$ and $g(Z_0) \subset X_0$. Assume that the fiber product $Y \times_X Z$ exists. Then $Y_0 \times_{X_0} Z_0$ exists and the natural morphism $Y_0 \times_{X_0} Z_0 \to Y \times_X Z$ is an open immersion. Indeed, let $p : Y \times_X Z \to Y$ and $q : Y \times_X Z \to Z$ be the natural projection. We may define $Y_0 \times_{X_0} Z_0$ as the open subscheme $p^{-1}(X_0) \cap q^{-1}(Y_0)$. It is easily checked that this subscheme satisfies the universal property. Note also that $Y_0 \times_{X_0} Z_0$ does not depends on X_0 containing $f(Y_0) \cup g(Z_0)$.

Step 2: Let $(X_i)_{i \in I}$ be an open covering of X. Set $Y_i = f^{-1}(X_i)$ and $Z_i = g^{-1}(X_i)$. Assume that the fiber products $Y_i \times_{X_i} Z_i$ exist. Then so does $Y \times_X Z$.

Indeed, by step 1, $T_{ij} = (Y_i \cap Y_j) \times_{X_i \cap X_j} (Z_i \cap Z_j)$ exists for all $(i,j) \in I^2$ and is naturally an open subscheme of $T_i = Y_i \times_{X_i} Z_i$. We also have natural isomorphisms $u_{ij} : T_{ij} \xrightarrow{\sim} T_{ji}$ satisfying to the condition of Lemma 2.48. Thus, we may glue the schemes T_i along the opens T_{ij} using the isomorphisms u_{ij} to get a scheme T. It is easy to check that T satisfies the universal property for fiber products.

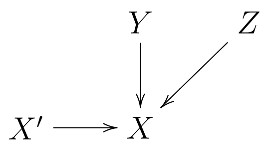
Step 3: Let $(Y_i)_{i \in I}$ be an open covering of Y. Assume that the fiber products $Y_i \times_X Z$ exist. Then so does $Y \times_X Z$. Indeed, by step 1, $(Y_i \cap Y_j) \times_X Z$ exists for all $(i,j) \in I^2$ and is naturally an open subscheme of $Y_i \times_X Z$. We also have isomorphisms $(Y_i \cap Y_j) \times_X Z \simeq (Y_j \cap Y_i) \times_X Z$. Using Lemma 2.48 we can glue the $Y_i \times_X Z$ along $(Y_i \cap Y_j) \times_X Z$ to obtain a scheme $Y \times_X Z$. One checks easily that this scheme satisfies the universal property for fiber products.

Step 4: Using step 2 and step 3 (for Y and Z), we may assume that X, Y and Z are affine schemes. The proposition follows then from Lemma 2.52. □

Remark 2.54 — If A is a ring there is a unique morphism $\mathbb{Z} \to A$. Using Proposition 2.25, we see that for any scheme X there is exactly one morphism $X \to \mathsf{Spec}(\mathbb{Z})$. The *direct product* $X \times Y$ of two schemes X and Y is defined to be the fiber product $X \times_{\mathsf{Spec}(\mathbb{Z})} Y$. We have a natural bijection $\hom(T, X \times Y) \simeq \hom(T,X) \times \hom(T,Y)$.

We often use fiber products of schemes as a black box, forgetting about their construction and retaining only their universal property. Here is an illustration of this principle.

Lemma 2.55 — ***1-*** *Suppose given a diagram of schemes*



There is a canonical isomorphism

$$X' \times_X (Y \times_X Z) \simeq (X' \times_X Y) \times_{X'} (X' \times_X Z).$$

2- Suppose given a cartesian square of schemes

$$\begin{array}{ccc} T & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

and a morphism of schemes $X' \to X$. *Then*

$$\begin{array}{ccc} X' \times_X T & \longrightarrow & X' \times_X Z \\ \downarrow & & \downarrow \\ X' \times_X Y & \longrightarrow & X' \end{array}$$

is also cartesian.

Proof. The first part follows from the second one. Let U be a scheme. We need to check that

$$\hom(U, X' \times_X T) \to \hom(U, X' \times_X Y) \times_{\hom(U,X')} \hom(U, X' \times_X Z)$$

is a bijection.

To give a morphism of schemes $U \to X' \times_X T$ is equivalent to give a couple of morphisms $U \to X'$ and $U \to T$ making commutative the square

$$\begin{array}{ccc} U & \longrightarrow & T \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

Using that T is the fiber product of Y and Z over X, this is also equivalent to give a triple of morphisms $U \to X'$, $U \to Y$ and $U \to Z$ such that

$$\begin{array}{ccccc} & & U & & \\ & \swarrow & \downarrow & \searrow & \\ Y & & X' & & Z \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

commutes. This is equivalent to give a commutative square

$$\begin{array}{ccc} U & \longrightarrow & X' \times_X Z \\ \downarrow & & \downarrow \\ X' \times_X Y & \longrightarrow & X'. \end{array}$$

This proves the lemma. □

We end this subsection with the following result:

Lemma 2.56 — *Let* $j : Y \to X$ *be an open (resp. closed, locally closed) immersion. Let* $f : X' \to X$ *be any morphism of schemes. Then the natural morphism* $Y \times_X X' \to X'$ *is an open (resp. closed, locally closed) immersion.*

Proof. The case of open immersions is clear (and has been already used). As a locally closed immersion may be factored as a closed immersion composed with an open one, we only need to consider the case of closed immersions.

The question is local on X and X'. Thus, we may assume that X and X' are affine. As j is a closed immersion, Y is also affine. We need to check that $\mathcal{O}_{X'}(X') \to \mathcal{O}_{Y\times_X X'}(Y \times_X X') = \mathcal{O}_Y(Y) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X'}(X')$ is surjective. Fix a tensor $s \otimes t$ with $s \in \mathcal{O}_Y(Y)$ and $t \in \mathcal{O}_{X'}(X')$. We may find a section $r \in \mathcal{O}_X(X)$ which maps to s by $\mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$. The we have $s \otimes t = (r.1_{\mathcal{O}_Y(Y)}) \otimes t = 1 \otimes rt$ which is in the image of $\mathcal{O}_{X'}(X') \to \mathcal{O}_Y(Y) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X'}(X')$. □

2.8. **Points and fiber products.**

Let X be a scheme and $x \in X$ a point. We may associate to x a morphism of schemes $\mathsf{Spec}(\kappa(x)) \to X$ as follows. Let U be an affine open subset of X containing x. Then the evaluation on x gives a morphism $\mathcal{O}_X(U) \to \kappa(x)$. Our morphism is then the composition

$$\mathsf{Spec}(\kappa(x)) \to \mathsf{Spec}(\mathcal{O}_X(U)) \simeq U \to X$$

which is clearly independent from the choice of the open affine neighborhood U. Note the following:

Lemma 2.57 — *The morphism $\mathsf{Spec}(\kappa(x)) \to X$ is a closed immersion if and only if x is a closed point of X (i.e., $\{x\}$ is a closed subset of X).*

Proof. The condition is sufficient. Indeed, if $\mathsf{Spec}(\kappa(x)) \to X$ is a closed immersion, its image (which is simply $\{x\}$) is a closed subset of X.

Now assume that $\{x\}$ is a closed subset of X. To check that $\mathsf{Spec}(\kappa(x)) \to X$ is a closed immersion, we may argue locally on X. Using the cover $X = U \cup (X\backslash\{x\})$ (with U an open affine neighborhood of x), we may assume that $X = \mathsf{Spec}(A)$ is affine and x is a prime ideal $\mathfrak{p} \subset A$. Then $\{\mathfrak{p}\}$ is closed in $\mathsf{Spec}(A)$ if and only if $\mathfrak{p}$ is maximal. In this case, $\kappa(\mathfrak{p}) = A/\mathfrak{p}$ and $A \to \kappa(\mathfrak{p})$ is surjective. This proves the lemma. □

The following proposition summaries some set theoretical properties of fiber products of schemes.

Proposition 2.58 — *Let $f : Y \to X$ and $g : Z \to X$ be morphisms of schemes. The obvious morphism*

$$\theta : |Y \times_X Z| \to |Y| \times_{|X|} |Z| \tag{5}$$

is surjective. More precisely, let $y \in Y$ and $z \in Z$ be such that $f(y) = g(z) = x$. Then the fiber of (5) at the point (y, z) is isomorphic to $\mathsf{Spec}(\kappa(y) \otimes_{\kappa(x)} \kappa(z))$ (which is not empty).

Proof. Set $T = Y \times_X Z$ and denote by $p : T \to Y$ and $q : T \to Z$ the obvious projections. We want to define a bijection

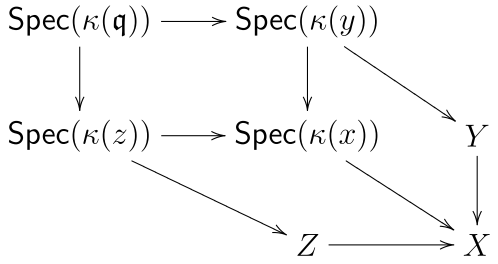
$$\theta^{-1}(y, z) \xrightarrow{\sim} \mathsf{Spec}(\kappa(y) \otimes_{\kappa(x)} \kappa(z)).$$

Let $t \in \theta^{-1}(y,z)$. Then we have a commutative square of fields extensions

$$\begin{array}{ccc} \kappa(x) & \longrightarrow & \kappa(y) \\ \downarrow & & \downarrow \\ \kappa(z) & \longrightarrow & \kappa(t) \end{array}$$

given a morphism of rings $\kappa(y) \otimes_{\kappa(x)} \kappa(z) \to \kappa(t)$. Taking the kernel of this morphism gives an element of $\mathsf{Spec}(\kappa(y) \otimes_{\kappa(x)} \kappa(z))$.

Reciprocally, given a prime ideal $\mathfrak{q} \in \mathsf{Spec}(\kappa(y) \otimes_{\kappa(x)} \kappa(z))$ we have the evaluation map $\kappa(y) \otimes_{\kappa(x)} \kappa(z) \to \kappa(\mathfrak{q})$. We may form the diagram



and get by the universal property of fiber products a map $\mathsf{Spec}(\kappa(\mathfrak{q})) \to Y \times_X Z$ whose image is a point of $\theta^{-1}(y,z)$. □

Definition 2.59 — *Let $f : Y \to X$ be a morphism of schemes and $x \in X$ a point. We define the (scheme-theoretic) fiber of f at x to be the scheme $Y_x = Y \times_X \mathsf{Spec}(\kappa(x))$.*

Corollary 2.60 — *The topological space $|Y_x|$ is naturally isomorphic to the subset $f^{-1}(x) \subset |X|$ with the induced topology. Moreover, when x is a closed point, the natural morphism $Y_x \to Y$ is a closed immersion.*

Proof. The last statement follows immediately from Lemmas 2.56 and 2.57.

By Proposition 2.58, the natural map $|Y_x| \to f^{-1}(x) = |Y| \times_{|X|} x$ is surjective and its fiber over $y \in f^{-1}(x)$ consists of $|\mathsf{Spec}(\kappa(y) \otimes_{\kappa(x)} \kappa(x))| \simeq |\mathsf{Spec}(\kappa(y))|$. As the latter contains exaclty one point, we deduce that $|Y_x| \to f^{-1}(x)$ is bijective.

It is clear that $|Y_x| \to f^{-1}(x)$ is continuous. We still need to check that it takes an open set to an open set. For this, we may assume that $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(B)$ are affine schemes and x correspond to a prime ideal $\mathfrak{p} \subset A$. In this case, $Y_{\mathfrak{p}} = \mathsf{Spec}(B \otimes_A \kappa(\mathfrak{p}))$. De only need to consider the case of a standard open subset $\mathrm{D}(f)$ for $f \in B \otimes_A \kappa(\mathfrak{p})$.

We have canonical isomorphisms of rings

$$B \otimes_A \kappa(\mathfrak{p}) = B \otimes_A (A/\mathfrak{p})_{\mathfrak{p}} \simeq B_{\mathfrak{p}} \otimes_A A/\mathfrak{p} \simeq B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}.$$

We may write the image of f by this composition as the class of a fraction $\left[\frac{u}{v}\right]$ where $u \in B$ and $v \in A \backslash \mathfrak{p}$. As $\mathrm{D}\left(\left[\frac{u}{v}\right]\right) = \mathrm{D}\left(\left[\frac{u}{1}\right]\right)$ we may assume that the image of f is of the form $\left[\frac{u}{1}\right]$. In other words, we may assume that $f = u \otimes 1 \in B \otimes_A \kappa(\mathfrak{p})$. But then, it is clear that $|Y_x| \to f^{-1}(x)$ sends $\mathrm{D}(f)$ to $f^{-1}(x) \cap \mathrm{D}(u)$. This finishes the proof of the lemma. □

2.9. **Separated morphisms.**

Let $f : Y \to X$ be a morphism of schemes. The commutative square

$$\begin{array}{ccc} Y & = & Y \\ \| & & \downarrow f \\ Y & \xrightarrow{f} & X \end{array}$$

gives a canonical morphism $\Delta_f : Y \to Y \times_X Y$ called the diagonal morphism. We have the following lemma:

Lemma 2.61 — *The morphism $\Delta_f : Y \to Y \times_X Y$ is a locally closed immersion.*

Proof. Let $y \in Y$ be a point. Let $V \subset Y$ be an open and affine neighborhood of y. We then have a commutative square

$$\begin{array}{ccc} V & \xrightarrow{\Delta_{f_{|V}}} & V \times_X V \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_f} & Y \times_X Y \end{array}$$

As $V \times_X V$ is naturally an open subscheme of $Y \times_X Y$, we see that it is sufficient to show that $V \to V \times_X V$ is a locally closed immersion. Thus, we may assume that Y is affine. Replacing y by a smaller open neighborhood, we may also assume that $f(Y)$ is contained in an affine open subset $X_0 \subset X$. As $Y \times_{X_0} Y \simeq Y \times_X Y$, we may thus assume that X is affine.

Now we are in the situation where $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(B)$ are affine. The diagonal $\mathsf{Spec}(B) \to \mathsf{Spec}(B \otimes_A B)$ correspond to the multiplication morphism

$$m : B \otimes_A B \to B$$

which sends a tensor $b \otimes b'$ to bb'. This morphism is clearly surjective, which means that $\mathsf{Spec}(B) \to \mathsf{Spec}(B \otimes_A B)$ is a closed immersion. □

Definition 2.62 — *A morphism of schemes $f : Y \to X$ is called* separated *if the diagonal morphism $\Delta_f : Y \to Y \times_X Y$ is a closed immersion. A scheme X is called* separated *if the morphism $X \to \mathsf{Spec}(\mathbb{Z})$ is separated.*

Lemma 2.63 — *Let $j : Y \to X$ be a locally closed immersion. Then $\Delta_j : Y \to Y \times_X Y$ is an isomorphism. In particular, j is separated.*

Proof. We first show that Δ_j is an open immersion. This is a local statement and we may assume that j is a closed immersion between affine schemes. Then the statement follows from the fact that $(A/\mathfrak{a}) \otimes_A (A/\mathfrak{a}) \to A/\mathfrak{a}$ is an isomorphism. To prove the lemma we still need to show that Δ_j is surjective. This follows immediately from Proposition 2.58. □

Lemma 2.64 —

1- *Let $f : Y \to X$ be a separated morphism and $g : X' \to X$ be any morphism of schemes. Then $Y \times_X X' \to X'$ is separated.*

2- *Let $f : Y \to X$ and $g : Z \to Y$ be two separated morphisms of schemes. Then $f \circ g$ is also separated.*

Proof. The first part follows from Lemma 2.56 and the fact that

$$Y \times_X X' \xrightarrow{\Delta} (Y \times_X X') \times_{X'} (Y \times_X X') = (Y \times_X Y) \times_X X'$$

is simply $\Delta_f \times_X X'$.

For the second part, we use the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\Delta_g"] \arrow[rr, bend left, "\Delta_{fg}"] & Z \times_Y Z \arrow[r, "d"] \arrow[d] & Z \times_X Z \arrow[d] \\ & Y \arrow[r, "\Delta_f"] & Y \times_X Y \end{tikzcd}$$

where the square is cartesian. Using Lemma 2.56, we see that Δ_{fg} is a closed immersion if Δ_g and Δ_f are closed immersions. □

Proposition 2.65 — *Let $j : Z \to Y$ and $f : Y \to X$ be morphisms of schemes. Assume that f is separated and $f \circ j$ is a closed immersion. Then j is a closed immersion.*

Proof. We have a cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_f"] \\ Z \times_X Y \arrow[r] & Y \times_X Y \end{tikzcd}$$

Indeed, let U be any scheme. To give a commutative square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_f"] \\ Z \times_X Y \arrow[r] & Y \times_X Y \end{tikzcd}$$

is equivalent to give morphisms $a : U \to Y$, $b : U \to Z$ and $c : U \to Y$ such that $a = c = b \circ j$. This is clearly equivalent to give the morphism $b : U \to Z$.

As f is separated, we have that Δ_f is a closed immersion. It follows that $Z \to Z \times_X Y$ is also a closed immersion. On the other hand, $Z \times_X Y \to X \times_X Y = Y$ is also closed immersion and j is the composition $Z \to Z \times_X Y \to Y$. This proves the proposition. □

Corollary 2.66 — *Let $f : Y \to X$ and $g : Z \to Y$ be morphisms of schemes. If $f \circ g$ is separated then so is g.*

Proof. We have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\Delta_g"] \arrow[rr, bend left, "\Delta_{fg}"] & Z \times_Y Z \arrow[r, "d"] \arrow[d] & Z \times_X Z \arrow[d] \\ & Y \arrow[r, "\Delta_f"] & Y \times_X Y \end{tikzcd}$$

with a cartesian square. As Δ_{fg} is a closed immersion and d is separated (being a locally closed immersion) we know by Proposition 2.65 that Δ_f is a closed immersion. □

Proposition 2.67 — *Let $f : Y \to X$ be a morphism of schemes. The following conditions are equivalent:*

(1) *f is separated,*
(2) *For every affine open subset $U \subset X$ and affine open subsets V_1, $V_2 \subset f^{-1}(U)$, the intersection $V_1 \cap V_2$ is affine and the natural morphism $\mathcal{O}_Y(V_1) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V_2) \to \mathcal{O}_Y(V_1 \cap V_2)$ which sends a tensor $s_1 \otimes s_2$ to $(s_1)_{|V_1\cap V_2}.(s_2)_{|V_1\cap V_2}$ is surjective.*
(3) *There exist an open covering $(U_i)_{i\in I}$ of X by affine subschemes and open coverings $(V_{ij})_{j\in J_i}$ of $f^{-1}(U_i)$ by affine subschemes such that $V_{ij} \cap V_{ij'}$ is affine and $\mathcal{O}_Y(V_{ij}) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_Y(V_{ij'}) \to \mathcal{O}_Y(V_{ij} \cap V_{ij'})$ is surjective for all $i \in I$ and j, $j' \in J_i$.*

Proof. We first show that (1) implies (2). Indeed, we may view $V_1 \cap V_2$ as the inverse image of the open $V_1 \times_U V_2 \subset Y \times_X Y$ by the diagonal morphism Δ_f. If Δ_f is a closed immersion, then so is $V_1 \cap V_2 \to V_1 \times_U V_2$. As the latter is affine, $V_1 \cap V_2$ is also affine and $\mathcal{O}_{V_1\times_U V_2}(V_1 \times_U V_2) \to \mathcal{O}_{V_1\cap V_2}(V_1 \cap V_2)$ is surjective.

It is clear that (2) implies (3). We still need to check that (3) implies (1). Indeed, we may cover the scheme $Y \times_X Y$ by the affine open subschemes $V_{ij} \times_{U_i} V_{ij'}$. Thus we only need to check that $\Delta_f(Y) \cap (V_{ij} \times_{U_i} V_{ij'})$ is a closed subscheme of $V_{ij} \times_{U_i} V_{ij'}$. As the latter is isomorphic to $V_{ij} \cap V_{ij'}$ which is affine, we only need to see that $\mathcal{O}_{V_{ij}\times_{U_i}V_{ij'}}(V_{ij} \times_{U_i} V_{ij'}) \to \mathcal{O}_{V_{ij}\cap V_{ij'}}(V_{ij} \cap V_{ij'})$ is surjective, which is the case by (3). □

2.10. The projective spectrum of a graded ring.

Recall that a grading on a ring A is a decomposition in direct factors $A = \oplus_{i\in\mathbb{Z}} A_i$ where $A_i \subset A$ are subgroups such that $A_i.A_j \subset A_{i+j}$ for all i, $j \in \mathbb{Z}$. A graded ring, is a ring with the choice of a grading. Every ring A can be graded in an obvious way by setting $A_0 = A$ and $A_i = 0$ for $i \neq 0$. An element $a \in A$ is called *homogenous* if it lies in one of the A_i. When a is non zero, then the i is unique and is called the *degree* of a.

Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a graded ring. A graded A-module is an A-module M together with a decomposition in direct factors $M = \oplus_{i\in\mathbb{Z}} M_i$ such that $A_i.M_j \subset M_{i+j}$ for all i, $j \in \mathbb{Z}$. Elements of M_i are called homogenous of degree i.

An $\mathbb{N}$-graded ring is a graded ring $A = \oplus_{i\in\mathbb{Z}} A_i$ such that $A_i = 0$ for $i < 0$. The purpose of this paragraph is to associate to an $\mathbb{N}$-graded ring $A = \oplus_{i\in\mathbb{N}} A_i$ a scheme $\mathsf{Proj}(A)$ which depends naturally on the grading. Typically, we will have that $\mathsf{Proj}(A) =$ when $A_i = 0$ for $i \neq 0$.

Definition 2.68 — *Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a graded ring. An ideal $I \subset A$ is called* homogenous *if it is generated by homogenous elements. This is equivalent to ask that $I = \oplus_{i\in\mathbb{N}}(A_i \cap I)$.*

Assume that $A = \oplus_{i\in\mathbb{N}} A_i$ is $\mathbb{N}$-graded. We define $\mathsf{Proj}(A)$ to be the set of prime homogenous ideals which do not contain the ideal $A_+ = \oplus_{i\geq 1} A_i$.

The purpose of this paragraph is to endow the set $\mathsf{Proj}(A)$ with the structure of a scheme.

The set $\mathsf{Proj}(A)$ is naturally a subset of $\mathsf{Spec}(A)$. The Zarisky topology on $\mathsf{Spec}(A)$ induces a topology on $\mathsf{Proj}(A)$ which we still call the Zariski topology. The Zariski closed subsets of $\mathsf{Proj}(A)$ are all of the form

$$\mathcal{Z}_+(E) = \{\mathfrak{p} \in \mathsf{Proj}(A),\ E \subset \mathfrak{p}\}$$

for a subset $E \subset A$. The subset $\mathcal{Z}_+(E)$ does not change when we replace E by the homogenous ideal generated by E (which is by definition the intersection of all homogenous ideals containing E).

The following lemma gives a way to construct homogenous prime ideals.

Lemma 2.69 — *Let $A = \oplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded ring. Given an ideal $\mathfrak{a} \subset A$ we set $\mathfrak{a}^h = \oplus_{i\in\mathbb{N}}(\mathfrak{a} \cap A_i)$. This is the homogenous ideal associated to $\mathfrak{a}$. If $\mathfrak{p}$ is a prime ideal then $\mathfrak{p}^h$ is a homogenous prime ideal.*

Proof. We argue by contradiction. Let $a,\ b \in A\backslash\mathfrak{p}^h$ such that $ab \in \mathfrak{p}^h$. We may write $a = \sum_{i=0^m} a_i$ and $b = \sum_{i=0}^m b_i$ with a_i and b_i homogenous of degree i. If $a_m \in \mathfrak{p}^h$, we may replace a by $a - a_m$ without changing our hypothesis. Thus, we may assume that $a_m \notin \mathfrak{p}^h$. Similarly, we may assume that $b_n \notin \mathfrak{p}^h$. But then

$$ab = a_m b_n + \sum_{i=0}^{m+n-1} c_i$$

where c_i are homogenous of degree i. As $ab \in \mathfrak{p}^h$, we deduce that $a_m b_n \in \mathfrak{p}^h \subset \mathfrak{p}$. Using that $\mathfrak{p}$ is prime, we may assume that $a_m \in \mathfrak{p}$. As a_m is homogenous, we actually have that $a_m \in \mathfrak{p}^h$. We have reached a contradiction. □

Proposition 2.70 — *Let $A = \oplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded ring and let $\mathfrak{a}$ and $\mathfrak{b}$ be homogenous ideals of A. Then $\mathcal{Z}_+(\mathfrak{a}) \subset \mathcal{Z}_+(\mathfrak{b})$ if and only if $\mathfrak{b} \cap A_+ \subset \sqrt{\mathfrak{a}}$.*

Proof. First, assume that $\mathfrak{b} \cap A_+ \subset \sqrt{\mathfrak{a}}$. Let $\mathfrak{p} \in \mathcal{Z}_+(\mathfrak{a})$. Then $\sqrt{\mathfrak{a}} \subset \mathfrak{p}$ which implies that $\mathfrak{b} \cap A_+ \subset \mathfrak{p}$. This means that $\mathfrak{p} \in \mathcal{Z}(\mathfrak{b}) \cup \mathcal{Z}(A_+)$. As $\mathsf{Proj}(A) \cap \mathcal{Z}(A_+) = \emptyset$, we get that $\mathfrak{p} \in \mathcal{Z}(\mathfrak{b})$ and thus $\mathfrak{p} \in \mathcal{Z}_+(\mathfrak{b})$.

Now assume that we have the inclusion $\mathcal{Z}_+(\mathfrak{a}) \subset \mathcal{Z}_+(\mathfrak{b})$. It is sufficient to show that $\mathcal{Z}(\mathfrak{a}) \subset \mathcal{Z}(\mathfrak{b} \cap A_+) = \mathcal{Z}(\mathfrak{b}) \cup \mathcal{Z}(A_+)$. So let $\mathfrak{p}$ be a prime ideal containing $\mathfrak{a}$. If $A_+ \subset \mathfrak{p}$ then $\mathfrak{p} \in \mathcal{Z}(A_+)$. So we may assume that $A_+ \not\subset \mathfrak{p}$. We clearly have $\mathfrak{a} \subset \mathfrak{p}^h$ and we know that $\mathfrak{p}^h$ is an homogenous prime ideal which does not contains A_+. It follows that $\mathfrak{p}^h \in \mathcal{Z}_+(\mathfrak{a}) \subset \mathcal{Z}_+(\mathfrak{b})$. It follows that $\mathfrak{b} \subset \mathfrak{p}^h \subset \mathfrak{p}$. We have shown that $\mathfrak{p} \in \mathcal{Z}(\mathfrak{b})$. The proposition is proven. □

Corollary 2.71 — *Let $A = \oplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded ring. Then $\mathsf{Proj}(A)$ is empty if and only if A_+ consists of nilpotent elements.*

Proof. Applying Proposition 2.70 to the inclusion $\mathcal{Z}_+(0) \subset \mathcal{Z}_+(A)$ we get that $A_+ \subset \sqrt{(0)}$. □

Remark 2.72 — Let $A = \oplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded ring. If $f \in A$ is a homogenous element, we write $\mathrm{D}_+(f)$ for the complement of $\mathcal{Z}_+(f)$ in $\mathsf{Proj}(A)$. This is a Zariski open subset of $\mathsf{Proj}(A)$. The family $(\mathrm{D}_+(f))_{f\in\cup_{i\geq 1} A_i}$ is a generating family of opens for the Zariski topology on $\mathsf{Proj}(A)$. Indeed, any open subset $U \in \mathsf{Proj}(A)$ is the complement of a subset of the form $\mathcal{Z}_+(\mathfrak{a})$ for a homogenous ideal $\mathfrak{a} \subset \mathcal{Z}_+$. We actually have $\mathcal{Z}_+(\mathfrak{a}) = \mathcal{Z}_+(\mathfrak{a} \cap A_+)$. Using the fact that $\mathfrak{a} \cap A_+$ is generated by its homogenous elements (of strictly positive degree) we get that

$$\mathcal{Z}_+(\mathfrak{a} \cap A_+) = \bigcap_{f \in \cup_{i \geq 1}(\mathfrak{a} \cap A_i)} \mathcal{Z}_+(f)$$

which translates to

$$U = \bigcup_{f \in \cup_{i \geq 1}(\mathfrak{a} \cap A_i)} \mathrm{D}_+(f).$$

To go further, we need to discuss fractions in a graded ring. Let A be a graded ring (not necessarily $\mathbb{N}$-graded). A multiplicative subset $S \subset A$ is called homogenous if it consists of homogenous elements. If S_i is the set of homogenous elements of degree i we have $S = \cup_{i \in \mathbb{Z}} S_i$ and $S_i.S_j \subset S_{i+j}$. Given a homogenous multiplicative subset $S \subset A$, the ring of fractions $S^{-1}A$ is naturally graded by

$$(S^{-1}A)_i = \left\{ \frac{a}{f},\ a \in A_{i+j} \text{ and } f \in S_j \right\}.$$

Moreover, if M is a graded A-module, $S^{-1}M$ is a graded $S^{-1}A$-module in a similar way.

Definition 2.73 — *A homogenous multiplicative subset S in a graded ring A is called* saturated *if for every homogenous element $a \in A$, the condition $ab \in S$ for some homogenous $b \in A$ implies that $a \in S$. Given an arbitrary homogenous multiplicative subset $S \subset A$, there is a smallest saturated homogenous multiplicative subset $\hat{S}_+$ namely, the set of all homogenous divisors of some element of S. Given a homogenous element $f \in A$, we define $\mathrm{S}'(f)$ to be the saturated homogenous multiplicative subset associated to $< f >$.*

Given two homogenous multiplicative subsets $S \subset T$ in a graded ring A, we have a morphism of graded rings $\rho_S^T : S^{-1}A \to S^{-1}T$. This morphism is an isomorphism if and only if $\hat{S}_+ = \hat{T}_+$.

Lemma 2.74 — *Assume that A is an $\mathbb{N}$-graded ring and let f and g be two homogenous elements of A of strictly positive degrees. Then $\mathrm{D}_+(f) \subset \mathrm{D}_+(g)$ if and only if $\mathrm{S}'(g) \subset \mathrm{S}'(f)$.*

Proof. If $\mathrm{S}'(g) \subset \mathrm{S}'(f)$ then $\mathrm{S}(g) \subset \mathrm{S}(f)$ and $\mathrm{D}(f) \subset \mathrm{D}(g)$. But, $\mathrm{D}_+(h) = \mathrm{D}(h) \cap \mathsf{Proj}(A)$ (as subsets of $\mathsf{Spec}(A)$) for any $h \in A$. This proves that our condition is sufficient.

To prove the necessity of our condition, we remark that $\mathrm{D}_+(f) \subset \mathrm{D}_+(g)$ implies by Proposition 2.76 that

$$\frac{g^{\deg(f)}}{f^{\deg(g)}}$$

is invertible in $(A_f)_0$. This easily implies that $g \in \mathrm{S}'(f)$. □

Let A be an $\mathbb{N}$-graded ring. Let M be a graded A-module (not necessarily $\mathbb{N}$-graded). We want to associate to M a sheaf for the Zariski topology on $\mathsf{Proj}(A)$. We first define a partial presheaf with respect to the generating family $(\mathrm{D}_+(f))_{f \in \cup_{i \geq 1} A_i}$.

Definition 2.75 — *We define a partial presheaf $\widetilde{M_+^{part}}$ on $\mathsf{Proj}(A)$ by sending $\mathrm{D}_+(f)$ to $(\mathrm{S}'(f)^{-1}M)_0$ and an inclusion $\mathrm{D}_+(f) \subset \mathrm{D}_+(g)$ to the canonical morphism*

$$\left(\rho_{\mathrm{S}'(g)}^{\mathrm{S}'(f)}\right)_0 : (\mathrm{S}'(g)^{-1}M)_0 \to (\mathrm{S}'(f)^{-1}M)_0$$

induced from the inclusion $\mathrm{S}'(g) \subset \mathrm{S}'(f)$.

We will see later that $\widetilde{M_+^{part}}$ is a partial sheaf on $\mathsf{Proj}(A)$. This will be a consequence of the following proposition:

Proposition 2.76 — *Let $f \in A$ be a homogenous element of strictly positive degree. There is a homeomorphism $\alpha_f : \mathrm{D}_+(f) \xrightarrow{\sim} \mathsf{Spec}((A_f)_0)$ which sends a homogenous prime ideal $\mathfrak{p} \subset A$ not containing f to the prime ideal $\mathfrak{p}_f \cap (A_f)_0$.*

Moreover, if $g \in A$ is another homogenous element, then α_f sends $\mathrm{D}_+(f) \cap \mathrm{D}_+(g)$ to

$$\mathrm{D}\left(\frac{g^{\deg(f)}}{f^{\deg(g)}}\right) \subset \mathsf{Spec}((A_f)_0).$$

Proof. We will freely use the fact that A_f is naturally graded (see the discussion after the proof of the proposition). We split the proof into several steps.
Step 1: Let $\mathfrak{q}_0 \subset (A_f)_0$ be a prime ideal. Set $\mathfrak{q} = \sqrt{\mathfrak{q}_0 A_f}$. We claim that $\mathfrak{q}$ is a graded prime ideal of A_f and that $\mathfrak{q} \cap (A_f)_0 = \mathfrak{q}_0$. The fact that $\mathfrak{q}$ is graded is an easy exercise. So is the last assertion.

To show that $\mathfrak{q}$ is prime, we argue by contradiction. Let $\frac{a}{f^m}$ and $\frac{b}{f^n}$ be two elements of $A_f \backslash \mathfrak{q}$ such that

$$\frac{ab}{f^{m+n}} \in \mathfrak{q}.$$

As $\frac{f}{1}$ is invertible in A_f, we may assume that $m = n = 0$. We can write $a = \sum_{i=u_1}^{v_1} a_i$ and $b = \sum_{i=u_2}^{v_2} b_i$ with $u_1 \leq v_1$ and $u_2 \leq v_2$. If $\frac{a_{u_1}}{1} \in \mathfrak{q}$ we may replace a by $a - a_{u_1}$ without changing our hypothesis. Thus we may assume that $\frac{a_{u_1}}{1} \notin \mathfrak{q}$ and similarly $\frac{b_{u_2}}{1} \notin \mathfrak{q}$. As

$$\frac{ab}{1} = \frac{a_{u_1}b_{u_2} + \sum_{i=u_1+u_2+1}^{v_1+v_2} c_i}{1} \in \mathfrak{q}$$

and using the fact that $\mathfrak{q}$ is homogenous, we deduce that

$$\frac{a_{u_1}b_{u_2}}{1} \in \mathfrak{q}.$$

Using the fact that $\frac{f}{1}$ is invertible in A_f and that $\deg(f) > 0$, we get that

$$\left(\frac{a_{u_1}^{\deg(f)}}{f^{u_1}}\right)\left(\frac{b_{u_2}^{\deg(f)}}{f^{u_2}}\right) \in \mathfrak{q} \cap (A_f)_0 = \mathfrak{q}_0.$$

Using the fact that $\mathfrak{q}_0$ is prime, we may assume that

$$\frac{a_{u_1}^{\deg(f)}}{f^{u_1}} \in \mathfrak{q}_0.$$

It follows that $\left(\frac{a_{u_1}}{1}\right)^{\deg(f)} \in \mathfrak{q}_0 A_f$. Thus, we get that $\frac{a_{u_1}}{1} \in \mathfrak{q}$. This is a contradiction.

Step 2: We now show that $\alpha_f : \mathrm{D}_+(f) \to \mathsf{Spec}((A_f)_0)$ is surjective. Let $\mathfrak{q}_0 \subset (A_f)_0$ be a prime ideal and $\mathfrak{q} = \sqrt{\mathfrak{q}_0 A_f}$ the homogenous prime ideal defined in the previous step. Let $\mathfrak{p} = (\rho^f)^{-1}(\mathfrak{q}) \subset A$ be the inverse image of $\mathfrak{q}$ along the canonical morphism $\rho^f : A \to A_f$. It is clear that $\mathfrak{p}$ is a homogenous prime ideal of A which does not contains f (and in particular, does not contains A_+). It remains to show that $\alpha_f(\mathfrak{p}) = \mathfrak{q}_0$, i.e., $\mathfrak{p}_f \cap (A_f)_0 = \mathfrak{q}_0$. This follows immediately from the fact that $\mathfrak{q} = \mathfrak{p}_f$ and the previous step.

To show the injectivity of α_f, we pick up two homogenous prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ not containing f. Assume that $\mathfrak{p}_f \cap (A_f)_0 = \mathfrak{p}'_f \cap (A_f)_0$. Now let $a \in \mathfrak{p}$ be a homogenous element. As

$$\frac{a^{\deg(f)}}{f^{\deg(a)}} \in \mathfrak{p}_f \cap (A_f)_0$$

it is also an element of $\mathfrak{p}'_f \cap (A_f)_0$. This implies that $a^{\deg(f)} \in \mathfrak{p}'$ and so $a \in \mathfrak{p}'$. This shows that $\mathfrak{p} \subset \mathfrak{p}'$ and by symmetry, we are done.

Step 3: The application $\alpha_f : \mathrm{D}_+(f) \to \mathsf{Spec}((A_f)_0)$ is continuous. Indeed, it can be written as the composition

$$\mathrm{D}_+(f) = \mathrm{D}(f) \cap \mathsf{Proj}(A) \subset \mathrm{D}(f) \simeq \mathsf{Spec}(A_f) \to \mathsf{Spec}((A_f)_0).$$

to show that α_f is a homeomorphism, we still need to check that α_f is open. This follows from the last assertion, which we now prove.

Let $\mathfrak{p}$ be an homogenous prime ideal not containing f nor g (which is equivalent to the condition that $\mathfrak{p}$ does not contains fg). Then $\mathfrak{p}_f$ does not contains $\frac{g^{\deg(f)}}{f^{\deg(g)}}$. This shows that

$$\alpha_f(\mathfrak{p}) = \mathfrak{p}_f \cap (A_f)_0 \in \mathrm{D}\left(\frac{g^{\deg(f)}}{f^{\deg(g)}}\right).$$

On the other hand, let $\mathfrak{q}_0 \subset (A_f)_0$ be a prime ideal not containing $\frac{g^{\deg(f)}}{f^{\deg(g)}}$. The homogenous prime ideal $\mathfrak{q} = \sqrt{\mathfrak{q}_0 A_f}$ does not contains $\frac{g^{\deg(f)}}{f^{\deg(g)}}$ neither. As $f1$ is invertible in A_f, we deduce that $\frac{g}{1} \notin \mathfrak{q}$. This shows that $g \notin \mathfrak{p} = (\rho^f)^{-1}(\mathfrak{q})$. We thus have constructed $\mathfrak{p} \in \mathrm{D}_+(f) \cap \mathrm{D}_+(g)$ such that $\alpha_f(\mathfrak{p}) = \mathfrak{q}_0$. This ends the proof of the proposition. ☐

Corollary 2.77 — *Let A be an $\mathbb{N}$-graded ring and M a graded A-module. Then $\widetilde{M}_+^{part}$ is a partial sheaf on* $\mathsf{Proj}(A)$. *More precisely, for a homogenous $f \in A_+$, the restriction of $\widetilde{M}_+^{part}$ to $\mathrm{D}_+(f)$ is isomorphic to $\alpha_f^*(\widetilde{(M_f)_0}^{part})$.*

Proof. It clearly suffices to show the second statement. Let $g \in A_+$ be a homogenous element such that $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$. We need to show that there is a canonical isomorphism

$$((M_f)_0)_{g^{\deg(f)} f^{-\deg(g)}} \xrightarrow{\sim} (M_g)_0$$

which is compatible to inclusions $\mathrm{D}_+(g') \subset \mathrm{D}_+(g)$. As $f \in \mathrm{S}'(g)$, there is a homogenous element $h \in A$ such that $fh = g^n$. In particular, $M_g \simeq (M_f)_{g^{\deg(f)} f^{-\deg(g)}}$. As $g^{\deg(f)} f^{-\deg(g)}$ is in $(A_f)_0$, we have

$$(M_g)_0 \simeq ((M_f)_{g^{\deg(f)} f^{-\deg(g)}})_0 \simeq ((M_f)_0)_{g^{\deg(f)} f^{-\deg(g)}}.$$

It is an easy exercise to check that this isomorphism is compatible to restrictions along $\mathrm{D}_+(g') \subset \mathrm{D}_+(g)$. □

As a corollary of this, we may extend $\widetilde{M}_+^{part}$ in a unique way to a sheaf of A_0-modules $\widetilde{M}_+$ on $\mathsf{Proj}(A)$. When $\widetilde{M} = A$ with its obvious grading, we obtain a sheaf of rings $\mathcal{O}_+ = \widetilde{A}_+$. It is easy to see that $\widetilde{M}_+$ is an $\mathcal{O}_+$-module in a natural way.

Theorem 2.78 — *Let $A = \oplus_{i \in \mathbb{N}} A_i$ be an $\mathbb{N}$-graded ring. The ringed space $(\mathsf{Proj}(A), \mathcal{O}_+)$ is a separated scheme. If M is a graded A-module, the $\mathcal{O}_+$-module $\widetilde{M}_+$ is a quasi-coherent $\mathcal{O}_+$-module.*

Proof. Everything follows immediately from Corollary 2.77 except that $\mathsf{Proj}(A)$ is separated. To check this, we apply the criterion of Proposition 2.67.

For $f, g \in A_+$ homogenous, $\mathrm{D}_+(f) \cap \mathrm{D}_+(g) = \mathrm{D}_+(fg)$ is affine. We still need to check that $(A_f)_0 \otimes_{\mathbb{Z}} (A_g)_0 \to (A_{fg})_0$ is surjective. We may assume for this that $\deg(f) = \deg(g)$. An element of $(A_{fg})_0$ is a fraction $\frac{a}{(fg)^n}$ with $\deg(a) = 2n.\deg(f)$. We can write this fraction as a product

$$\frac{a}{(fg)^n} = \frac{a}{f^{2n}} \times (\frac{f}{g})^n.$$

The result is now clear. □

Example 2.79 — Let A be an $\mathbb{N}$-graded ring and M be a graded A-module. For $n \in \mathbb{Z}$, we let $M(n)$ be the A-module M graded by $M(n)_i = M_{i+n}$. Of particular interest, is the module $A(n)$. The quasi-coherent $\mathcal{O}_+$-module $\widetilde{A(n)}_+$ will be denoted by $\mathcal{O}_+(n)$.

Let A be an $\mathbb{N}$-graded module. Let M and N be two graded A-module. The A-module $M \otimes_A N$ can be graded in a natural way by setting $(M \otimes_A N)_n$ to be the subgroup generated by tensors $a \otimes b$ with $\deg(a) + \deg(b) = n$. Indeed, we know that $M \otimes_{\mathbb{Z}} N = \oplus_{i,j \in \mathbb{Z}} M_i \otimes N_j$. Thus, $M \otimes_{\mathbb{Z}} N$ is a $\mathbb{Z}$-graded group. The group $(M \otimes_A N)$ is defined as the quotient of $(M \otimes_{\mathbb{Z}} N)$ by the elements of the form $ax \otimes y - x \otimes ay$. As these elements are homogenous, we see that $M \otimes_A N = \oplus_{n \in \mathbb{Z}} (M \otimes_A N)_n$.

Proposition 2.80 — *Let A be an $\mathbb{N}$-graded ring and M and N two graded A-modules. There is a natural morphism*

$$\widetilde{M}_+ \otimes_{\mathcal{O}_+} \widetilde{N}_+ \to \widetilde{M \otimes_A N}. \tag{6}$$

Moreover, if the ideal A_+ is generated by A_1, (6) is invertible.

Proof. If $f \in A_+$ is a homogenous element, we define (6) over $\mathrm{D}_+(f)$ by

$$(M_f)_0 \otimes_{(A_f)_0} (N_f)_0 \to ((M \otimes_A N)_f)_0. \tag{7}$$

This easily checked to give a morphism of $\mathcal{O}_+$-modules.

Assume now that A_+ is generated by A_1. Then $\mathsf{Proj}(A) = \cup_{f \in A_1} \mathrm{D}_+(f)$. Thus we need to show that for $f \in S_1$, (7) is invertible.

Using the fact that f is of degree 1, we may define a morphism of rings $A \to (A_f)_0$ by sending a fraction homogenous elements $g \in A$ to $\frac{g}{f^{\deg(g)}}$. This morphism maps f to 1 and thus induces a morphism of rings $A_f \to (A_f)_0$. In particular, any $(A_f)_0$-module is naturally an A_f-module.

Now, given a graded A-module L, we claim that $(L_f)_0$ is canonically isomorphic (as an $(A_f)_0$-module) to $L_f \otimes_{A_f} (A_f)_0$. Indeed, multiplication by f induces isomorphisms $(L_f)_n \to (L_f)_{n+1}$ for all $n \in \mathbb{Z}$. Thus dividing by the ideal $(f-1)$ identifies $(L_f)_n$ with $(L_f)_{n+1}$.

We now return to the morphism (7). We have natural isomorphisms

$$(M_f)_0 \otimes_{(A_f)_0} (N_f)_0 \simeq (M_f \otimes_{A_f} (A_f)_0) \otimes_{(A_f)_0} (N_f \otimes_{A_f} (A_f)_0)$$

$$\simeq (M_f \otimes_{A_f} N_f) \otimes_{A_f} (A_f)_0 \simeq (M \otimes_A N)_f \otimes_{A_f} (A_f)_0 \simeq ((M \otimes_A N)_f)_0.$$

We leave it as an exercise to check that the above composition is equal to (7). □

Proposition 2.81 — *Assume that the ideal A_+ is generated by A_1. Then $\mathcal{O}_+(n)$ is locally free $\mathcal{O}_+$-module of rank one. Moreover, there are natural isomorphisms $\mathcal{O}_+(m) \otimes \mathcal{O}_+(n) \simeq \mathcal{O}_+(m+n)$.*

Proof. The second assertion follows from Proposition 2.80. We only prove the first assertion. As $\mathsf{Proj}(A) = \bigcup_{f \in A_1} \mathrm{D}_+(f)$ we only need to check that the $(A_f)_0$-module $(A_f)_n$ is free of rank 1 for all $f \in A_1$. But it is obvious that the map $(A_f)_0 \to (A_f)_n$ sending $\frac{a}{f^{\deg(a)}}$ to $\frac{a}{f^{\deg(a)-n}}$ is an isomorphism of $(A_f)_0$-modules. This proves the proposition. □

The following lemma shows that the condition A_+ generated by A_1 can often (for instance, if A_+ is generated by A_n for $n > 1$) be assumed after a mild modification of the graded ring A that does not affect the scheme $\mathsf{Proj}(A)$.

Lemma 2.82 — *Let A be an N-graded ring. For $n \in \mathbb{N} - \{0\}$ we set $A^{(n)} = \oplus_{d \in \mathbb{N}} A_{dn}$ graded by $A^{(n)}_d = A_{dn}$. Then, there is a canonical isomorphism $\mathsf{Proj}(A) \simeq \mathsf{Proj}(A^{(n)})$.*

Proof. This follows immediately from the fact that for $f \in A_+$ homogenous, $(A_f)_0 \simeq (A^{(n)}_{f^n})_0$. □

The following result describes fairly good quasi-coherent modules on $\mathsf{Proj}(A)$.

Proposition 2.83 — *Let $A = \oplus_{n \in \mathbb{N}} A_n$ be an $\mathbb{N}$-graded ring such that ideal A_+ is generated by A_1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_+$-module. We set $\Gamma^+_n(\mathcal{F}) = (\mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(n))(\mathsf{Proj}(A))$ to be the global section of $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(n)$. Then $\Gamma^+(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \Gamma^+_n(\mathcal{F})$ is naturally a graded A-module. Moreover, there is a natural isomorphism*

$$\widetilde{\Gamma^+(\mathcal{F})} \to \mathcal{F}. \tag{8}$$

Finally, if A_+ is generated by finitely many elements in A_1, then (8) is invertible.

Proof. It is clear that $\Gamma^+ = \oplus_{n\in\mathbb{Z}}\mathcal{O}_+(n)(\mathsf{Proj}(A))$ is a commutative ring. Moreover, we have a graded homomorphism of rings $A \to \Gamma^+$ sending $a \in A_n$ to $\dfrac{fa}{f} \in \mathcal{O}_+(n)(\mathrm{D}(f))$ for all $f \in A_+$ homogenous. As $\Gamma^+(\mathcal{F})$ is obviously a graded Γ^+-module, it is also a graded A-module.

We define the morphism $\widetilde{\Gamma^+(\mathcal{F})} \to \mathcal{F}$ as follows. If $f \in A_1$, an element of $(\Gamma^+(\mathcal{F})_f)_0$ is a fraction $\dfrac{s}{f^r}$ with $s \in \Gamma^+_r(\mathcal{F})$ a global section of $\mathcal{F}(r)$. The restriction of s to $\mathrm{D}_+(f)$ is an element of $\mathcal{F}(\mathrm{D}_+(f)) \otimes_{(A_f)_0} (A_f)_r$. Thus, it can be uniquely written as a tensor $t \otimes \frac{f^r}{1}$ with $t \in \mathcal{F}(\mathrm{D}_+(f))$. Our morphism associates to $\dfrac{s}{f^r}$ the section t.

Now assume that A_+ is generated by finitely many elements $g_1, \ldots, g_n$ of A_1. To see that (8) is invertible, we need to check that

$$(\Gamma^+(\mathcal{F})_f)_0 \to \mathcal{F}(\mathrm{D}_+(f)) \tag{9}$$

is invertible for all $f \in A_1$.

We first prove surjectivity. We have canonical isomorphisms

$$\mathcal{F}(\mathrm{D}_+(g_i))_{\frac{f}{g_i}} \simeq \mathcal{F}(\mathrm{D}_+(f) \cap \mathrm{D}_+(g_i)) \simeq \mathcal{F}(\mathrm{D}_+(f))_{\frac{g_i}{f}}$$

Thus, there exists $t_i \in \mathcal{F}(\mathrm{D}_+(g_i))$ and $e_i \geq 0$ such that $t_{|\mathrm{D}_+(fg_i)} = (\frac{f}{g_i})^{-e_i}\ t_{i|\mathrm{D}_+(fg_i)}$. We may assume that e is independent of the i (after replacing t_i by $(\frac{f}{g_i})^{e-e_i}t_i$ for e bigger that all the e_i).

Now form the sections $s \otimes f^e \in (\mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(e))(\mathrm{D}_+(f))$ and $t_i \otimes g_i^e \in (\mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(e))(\mathrm{D}_+(g_i))$. Then, $(s \otimes f^e)_{|\mathrm{D}_+(fg_i)} = (t_i \otimes g_i^e)_{|\mathrm{D}_+(fg_i)}$. Moreover, there exists r big enough such that $(\frac{f}{g_i})^r(t_i - (\frac{g_j}{g_i})^e t_j) = 0$. This means that $(s \otimes f^{r+e})$ and $t_i \otimes f^r g_i^e$ form a locally defined section of the sheaf $\mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(r+e)$. Call $v \in \mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(r+e)(\mathsf{Proj}(A))$ the induced global section. Then (9) sends $\frac{v}{f^{r+e}}$ to s.

To prove injectivity, let $t \in (\mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(n))(\mathsf{Proj}(A))$ be a section such that $\frac{t}{f^n}$ is sent to zero by (9). This means that $t_{|\mathrm{D}_+(f)} = 0$. We need to prove that for r big enough, $t \otimes f^r(\mathcal{F} \otimes_{\mathcal{O}_+} \mathcal{O}_+(n+r))(\mathsf{Proj}(A))$ is zero. It is sufficient to have $(\frac{f^{r_i}}{g_i^{r_i}})t_{|\mathrm{D}_+(g_i)} = 0$ in $\mathcal{F}(\mathrm{D}_+(g_i))$ for each i. This is equivalent to ask that $t_{|\mathrm{D}_+(g_i)}$ becomes zero over $\mathrm{D}_+(g) \cap \mathrm{D}_+(f)$ which is clear. □

We end this paragraph by the following useful result:

Proposition 2.84 — *Let $\theta : A \to B$ be a morphism of $\mathbb{N}$-graded ring. Assume that the ideal B_+ is generated by $\theta(A_+)$. Then θ induces a morphism of schemes* $\mathsf{Proj}(B) \to \mathsf{Proj}(A)$.

Proof. Let $\mathfrak{q} \subset B$ be a graded prime ideal not containing B_+. As $\theta(A_+)$ generates B_+, we have $\theta(A_+) \not\subset \mathfrak{q}$ so that $\theta^{-1}(\mathfrak{q})$ do not contains A_+. Moreover, $\theta^{-1}(\mathfrak{q})$ is a graded prime ideal. This gives a continuous map $\alpha : |\mathsf{Proj}(B)| \to |\mathsf{Proj}(A)|$. Moreover, $\alpha^{-1}(\mathrm{D}_+(f)) = \mathrm{D}_+(\theta(f))$ for all homogenous $f \in A_+$.

To define a morphism of schemes, we still need to define the morphism of sheaves $\mathcal{O}_+ \to \alpha_*(\mathcal{O}_+)$. It is sufficient to define a morphism on the associated partial sheaves, i.e., to specify a compatible family of morphism of rings

$$\mathcal{O}_+(\mathrm{D}_+(f)) \to \mathcal{O}_+(\mathrm{D}_+(\alpha(f))).$$

We take the natural morphism $(A_f)_0 \to (B_{\alpha(f)})_0$. □

2.11. **The projective space.**

Let A be a ring and $t_0, \dots, t_n$ indeterminates. For $\underline{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$, we set $\underline{t}^{\underline{i}} = t_0^{i_0} \dots t_n^{i_n}$. The integer $|\underline{i}| = i_0 + \dots + i_n$ is called the total degree of the monomial $\underline{t}^{\underline{i}}$. The ring $A[\underline{t}] = A[t_0, \dots, t_n]$ admits a natural grading such that $A[\underline{t}]_d$ is the free A-module generated by the monomials $\underline{t}^{\underline{i}}$ of total degree d.

Definition 2.85 — *The scheme* $\mathsf{Proj}(A[t_0, \dots, t_n])$ *is called the* projective space *of dimension n over A and will be denoted by $\mathbb{P}^n_A$.*

The ideal $A[\underline{t}]_+$ is generated by $t_0, \dots, t_n$. It follows that the scheme $\mathbb{P}^n_A$ can be covered by $n+1$ affine schemes $\mathrm{D}_+(t_i)$ for $0 \le i \le n$.

Lemma 2.86 — *The affine scheme $\mathrm{D}_+(t_i)$ is isomorphic to* $\mathsf{Spec}(A[u_1, \dots, u_n])$ *where u_j are unknowns.*

Proof. Indeed, $\mathrm{D}_+(t_i)$ is canonically isomorphic to $\mathsf{Spec}((A[\underline{t}]_{t_i})_0)$. An element $\frac{P}{t_i^r} \in A[\underline{t}]_{t_i}$ is of degree zero if and only if the polynomial P is homogenous of degree r. Thus, we can write P as a finite sum $\sum_{|\underline{e}|=r} a_{\underline{e}} \underline{t}^{\underline{e}}$. We then get

$$\frac{P}{t_i^r} = \sum_{|\underline{e}|=r} a_{\underline{e}} \prod_{j=0}^{n} \left(\frac{t_j}{t_i}\right)^{e_j} = \sum_{|\underline{e}|=r} a_{\underline{e}} \prod_{j \neq i} \left(\frac{t_j}{t_i}\right)^{e_j}.$$

This shows that $(A[\underline{t}]_{t_i})_0$ is canonically isomorphic to the ring

$$A\left[\frac{t_j}{t_i};\ j \in [\![0,n]\!] \backslash \{i\}\right]$$

which can be identified with a ring of polynomials in n variables. □

Remark 2.87 — The scheme $\mathsf{Spec}(A[u_1, \dots, u_n])$ is called the *affine space* of dimension n over A and will be denoted by $\mathbb{A}^n_A$. We have shown that the projective space $\mathbb{P}^n_A$ can be covered by $n+1$ copies of the affine space $\mathbb{A}^n_A$.

Proposition 2.88 — *The canonical morphism*

$$A[\underline{t}]_d \to \mathcal{O}_{\mathbb{P}^n_A}(d)(\mathbb{P}^n_A)$$

is an isomorphism of A-module.

Proof. By definition $\mathcal{O}_{\mathbb{P}^n_A}(d)(\mathbb{P}^n_A)$ is the set of $n+1$-tuples $(P_i)_{i=0,\dots,n}$ where $P_i \in (A[\underline{t}]_{t_i})_d$ such that $P_i = P_j$ as elements of $(A[\underline{t}]_{t_i t_j})_d$.

We may write P_i as a fraction $\frac{Q_i}{t_i^r}$ with Q_i homogenous of degree $r+d$ and r independent of i.

For $i \neq j$, we have $t_j^r Q_i = t_i^r Q_j$. This clearly implies that t_j divides Q_j. In other words, P_i is a homogenous polynomial of degree d and $P_i = P_j$. This proves the proposition. □

Proposition 2.89 — *There is a natural isomorphism* $\mathbb{P}^n_A \simeq \mathbb{P}^n_{\mathbb{Z}} \times \mathsf{Spec}(A)$.

Proof. Any element $a \in A$ can be considered as an element of $A[\underline{t}]$ of degree 0. Thus, $\dfrac{a}{t_i^0}$ is a section of $(A[\underline{t}]_{t_i})_0 = \mathcal{O}_+(\mathrm{D}(t_i))$ for all i. These sections glue to give a section $a \in \mathcal{O}_+(\mathbb{P}^n_A)$. This gives a morphism of rings $A \to \mathcal{O}_+(\mathbb{P}^n_A)$ and hence a morphism of schemes $\mathbb{P}^n_A \to \mathsf{Spec}(A)$.

On the other hand, we have a morphism of schemes $\mathbb{P}^n_A \to \mathbb{P}^n_{\mathbb{Z}}$ given by ??. This show that we have a morphism of schemes $\mathbb{P}^n_A \to \mathbb{P}^n_{\mathbb{Z}} \times \mathsf{Spec}(A)$. To show that this is an isomorphism, we may use Lemma 2.86 to reduce to a similar statement for affine spaces. The result follows then from the canonical isomorphism $\mathbb{Z}[u_1, \ldots, u_n] \otimes A \simeq A[u_1, \ldots, u_n]$. ☐

Let X be a scheme. We want to describe the morphisms of schemes from X to $\mathbb{P}^n_A$. By Proposition 2.89 we have:

$$\hom(X, \mathbb{P}^n_A) \simeq \hom(X, \mathbb{P}^n_{\mathbb{Z}}) \times \hom(X, \mathsf{Spec}(A)) \simeq \hom(X, \mathbb{P}^n_{\mathbb{Z}}) \times \hom(A, \mathcal{O}_X(X)).$$

So it is sufficient to consider the case of $A = \mathbb{Z}$.

Before stating our main theorem, we need to discuss inverse images of quasi-coherent sheaves. Let $f : Y \to X$ be a morphism of schemes and $\mathcal{N}$ an $\mathcal{O}_X$-module. We define $f^\star\mathcal{N}$ to be the sheaf associated to the presheaf

$$V \subset Y \qquad \rightsquigarrow \qquad f^*\mathcal{N}(V) \otimes_{f^*\mathcal{O}_X(V)} \mathcal{O}_Y(V).$$

When $j : U \to X$ is the inclusion of an open subscheme, $j^\star(\mathcal{N})$ is naturally isomorphic to the restriction of $\mathcal{N}$ to U. Given another morphism of schemes $g : Z \to Y$, there is a natural isomorphism $(f \circ g)^\star\mathcal{N} \simeq g^\star f^\star\mathcal{N}$.

If $\mathcal{N}$ is a quasi-coherent $\mathcal{O}_X$-module then $f^\star\mathcal{N}$ is also quasi-coherent. Indeed, the question is local on X and Y. Thus, we may assume that $X = \mathsf{Spec}(A)$, $Y = \mathsf{Spec}(B)$ and $\mathcal{N} = \widetilde{N}$ for an A-module N. Then f corresponds to a morphism of rings $A \to B$ and $f^\star\widetilde{N} = \widetilde{N \otimes_A B}$. This follows immediately from the definition.

Note also that $f^\star\mathcal{O}_X \simeq \mathcal{O}_Y$. We now can state our main result for this paragraph:

Theorem 2.90 — *Let X be a scheme. There is a bijection from the set $\hom(X, \mathbb{P}^n_{\mathbb{Z}})$ to the set of quasi-coherent $\mathcal{O}_X$-submodules $\mathcal{H} \subset \mathcal{O}_X^{n+1}$ such that the quotient $\mathcal{O}_X^{n+1}/\mathcal{H}$ is locally free of rank 1. Moreover, this bijection is natural in X in the following sense. Let $f : Y \to X$ be a morphism of schemes and $a : X \to \mathbb{P}^n_{\mathbb{Z}}$ corresponding to $\mathcal{N} \subset \mathcal{O}_X^n$. Then $a \circ f$ corresponds to $f^\star\mathcal{N}$ considered as an $\mathcal{O}_Y$-submodule of $\mathcal{O}_Y^n$.*

Proof. We split the proof into several steps. The first one is a reduction to the case where X is affine.

Step 1: Let $(X_i)_{i\in I}$ be an open covering of X and assume that the theorem is true for X_i and $X_i \cap X_j$ for all $i, j \in I$. Then the theorem is also true for X.

First, let $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ be a morphism of schemes. Then the restrictions $f_i : X_i \to \mathbb{P}^n_{\mathbb{Z}}$ determines submodules $\mathcal{N}_i \subset \mathcal{O}_{X_i}^{n+1}$. Moreover, as f_i and f_j are equals on $X_i \cap X_j$, we have that $\mathcal{N}_{i|X_i\cap X_j} = \mathcal{N}_{j|X_i\cap X_j}$. Thus we may glue the subsheaves $\mathcal{N}_i \subset \mathcal{O}_{X_i}^{n+1}$ to get a $\mathcal{O}_X$-submodule $\mathcal{N} \subset \mathcal{O}_X^{n+1}$ such that $\mathcal{N}_{|X_i} = \mathcal{N}_i$. Moreover, $\mathcal{N}$ is quasi-coherent (as it is the case for each $\mathcal{N}_i$) and $\mathcal{O}_X^{n+1}/\mathcal{N}$ is locally free of rank 1 (as it is the case for each $\mathcal{O}_{X_i}^{n+1}/\mathcal{N}_i$).

Next, assume we are given a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{N} \subset \mathcal{O}_X^{n+1}$ such that $\mathcal{O}_X/\mathcal{N}$ is locally free. Define $\mathcal{N}_i = \mathcal{N}_{|X_i}$. Then $\mathcal{N}_i$ induces a morphism of schemes

$f_i : X_i \to \mathbb{P}^n_{\mathbb{Z}}$. As $\mathcal{N}_{i|X_i\cap X_j} = \mathcal{N}_{j|X_i\cap X_j}$ we deduce that f_i and f_j coincides on $X_i \cap X_j$. Thus, we may glue the f_i to get a morphism of schemes $f : X \to \mathbb{P}^n_{\mathbb{Z}}$.

It is now easy to reduce to the case where X is affine. Let $(X_i)_{i\in I}$ be an open covering of X by affine subschemes. When X is separated, $X_i \cap X_j$ is also affine so we may apply the previous discussion to conclude. In the general case, $X_i \cap X_j$ is an open subscheme of X_i. In particular, it is separated and the theorem holds for $X_i \cap X_j$. Thus we are done in this case too.

Step 2: Now we assume that $X = \mathsf{Spec}(A)$ is affine. We let $\mathscr{H}$ be the set of submodules $N \subset A^{n+1}$ such that A^{n+1}/N is locally free of rank one. This is the set of hyperplanes in A^{n+1}. In this step we define an application

$$\omega : \mathscr{H} \to \hom(\mathsf{Spec}(A), \mathbb{P}^n_{\mathbb{Z}}).$$

Let $N \subset A^{n+1}$ be an element of $\mathscr{H}$. Call $E = A^{n+1}/N$. We define a graded ring

$$A[E] = A \oplus E \oplus E^{\otimes 2} \oplus \cdots \oplus E^{\otimes n} \oplus \ldots$$

This ring is commutative. Indeed, if $f \in A$ the A-algebra $A[E]_f$ is canonically isomorphic to $A_f[E_f]$. Moreover, if $E_f \simeq A_f$, we get $A_f[E_f] \simeq A_f[t]$.

Let's denote $e_0, \ldots, e_n$ the canonical basis of the A-module A^{n+1}. We have a morphism of graded rings

$$\theta : \mathbb{Z}[t_0, \ldots, t_n] \to A[E]$$

sending t_i to the image of e_i in E. As E is a quotient of A^{n+1}, we see that $\theta(\mathbb{Z}[\underline{t}]_+)$ generates the ideal $A[E]_+$. By Proposition 2.84 we have a morphism $\mathsf{Proj}(A[E]) \to \mathsf{Proj}(\mathbb{Z}[\underline{t}]) = \mathbb{P}^n_{\mathbb{Z}}$. It remains to show that $\mathsf{Proj}(A[E])$ is canonically isomorphic to $\mathsf{Spec}(A)$.

We have a canonical morphism $\mathsf{Proj}(A[E]) \to \mathsf{Spec}(A)$ as A maps naturally to $\mathcal{O}_+(\mathsf{Proj}(A[E]))$. We claim that this morphism is invertible. To prove this, we may argue locally on $\mathsf{Spec}(A)$. Thus we may replace A and E by A_f and E_f for f varying in a set of generators of the ideal A. In particular, we may assume that E is free of rank one, i.e., we are reduced to show that $\mathsf{Proj}(A[t]) \to \mathsf{Spec}(A)$ is invertible. But, as t generates the ideal $A[t]_+$, we have that $\mathsf{Proj}(A[t]) = \mathrm{D}_+(t) \simeq \mathsf{Spec}(A[t, t^{-1}]_0) = \mathsf{Spec}(A)$. This proves our claim.

Step 3: We now define an application

$$\nu : \hom(\mathsf{Spec}(A), \mathbb{P}^n_{\mathbb{Z}}) \to \mathscr{H}.$$

Recall that we have a locally free $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}$-module $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) = \widetilde{\mathbb{Z}[\underline{t}](1)}$. Moreover, we have a natural morphism of $\mathbb{Z}$-modules

$$\mathbb{Z}[\underline{t}]_1 \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)(\mathbb{P}^n_{\mathbb{Z}})$$

which induces a morphism of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}$-modules

$$c : \mathcal{O}^{n+1}_{\mathbb{P}^n_{\mathbb{Z}}} \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1).$$

The morphism c is surjective. This is a local statement. But, over $\mathrm{D}_+(t_i)$, the module $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$ is generated by the section $\dfrac{t_i}{t_i^0} \in (\mathbb{Z}[\underline{t}]_{t_i})_1$ which is the image by c of the element e_i of the canonical basis of $\mathcal{O}^{n+1}_{\mathbb{P}^n_{\mathbb{Z}}}$.

We define $\mathcal{T} = \ker(c)$. It is easy to check that $\mathcal{T}$ is locally free of rank n by looking at the restriction of c over $\mathrm{D}_+(t_i)$.

Given a morphism of schemes $f : \mathsf{Spec}(A) \to \mathbb{P}^n_{\mathbb{Z}}$, we may consider $f^\star(\mathcal{T})$. This is a quasi-coherent $\mathcal{O}_{\mathsf{Spec}(A)}$-module. Moreover, it is naturally a submodule of $\mathcal{O}^{n+1}_{\mathsf{Spec}(A)}$. To check this, we may work locally on $\mathsf{Spec}(A)$ and more precisely, relatively to the cover $f^{-1}(\mathrm{D}_+(t_i))$. In particular, we may assume that the image of f is contained in $\mathrm{D}_+(t_i)$. Then the claim is trivial as $\mathcal{T}_{|\mathrm{D}_+(t_i)}$ is a direct factor of $\mathcal{O}^{n+1}_{\mathrm{D}_+(t_i)}$.

Now, let $N = f^\star\mathcal{T}(\mathsf{Spec}(A))$ and considered as a submodule of A^{n+1}. The quotient $E = A^{n+1}/N$ is locally free of rank one. Indeed, $\widetilde{E}$ is canonically isomorphic to $f^\star(\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1))$. Thus, we get in this way an element of $\mathscr{H}$ which is our $\nu(f)$.

Step 4: Let's check that $\omega\circ\nu = \mathrm{id}$. Let $f : \mathsf{Spec}(A) \to \mathbb{P}^n_{\mathbb{Z}}$ be a morphism of schemes. Let $N = f^\star(\mathcal{T}) \subset A^{n+1}$ and $E = A^{n+1}/N$. We need to check that f coincides with the canonical morphism $\mathsf{Proj}(A[E]) \to \mathbb{P}^n_{\mathbb{Z}}$ induced from $\mathbb{Z}[\underline{t}] \to A[E]$ modulo the isomorphism $\mathsf{Spec}(A) \simeq \mathsf{Proj}(A[E])$.

The question is local on $\mathsf{Spec}(A)$. Thus, we may assume that f factors trough $\mathrm{D}_+(t_0) \simeq \mathsf{Spec}(\mathbb{Z}[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}])$. Let $\tau_i = \frac{t_i}{t_0} \circ f$.

We have a canonical generator $t_0 \in \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)(\mathrm{D}_+(t_0))$. Denote by s_0 the image of t_0 in E. This is a generator of the A-module E. The graded morphism $\mathbb{Z}[t_0, \dots, t_n] \to A[E] = A[s_0]$ sends t_0 to s_0 and t_i to $\tau_i s_0$ for $1 \le i \le n$. Thus, it sends $\frac{t_i}{t_0}$ to τ_i.

Step 5: To end the proof of the theorem we still need to check that $\nu\circ\omega = \mathrm{id}$. Let $N \subset A^{n+1}$ be a submodule such that A^{n+1}/N is locally free. Call $f : \mathsf{Spec}(A) \simeq \mathsf{Proj}(A[E]) \to \mathbb{P}^n_{\mathbb{Z}}$ the morphism $\omega(N)$. We need to check that $N = f^\star\mathcal{T}(\mathsf{Spec}(A))$.

Here also the question is local over $\mathsf{Spec}(A)$ as we want to check that two submodule of A^{n+1} are equals. Thus, we may assume that $A^{n+1}/N = E$ is free of rank 1. We choose an isomorphism $E \simeq A$ and call $l : A^{n+1} \to A$ the obvious map. This is a linear form and can be written as $l(x_0, \dots, x_n) = a_0x_0 + \dots + a_nx_n$.

It is clear that the ideal generated by the a_i is A itself. In other words, we have $\mathsf{Spec}(A) = \cup_{i=0}^n \mathrm{D}(a_i)$. Replacing A by one of the A_{a_i} and permuting the variables, we may assume that a_0 is invertible in A. Replacing the a_i by $a_0^{-1}a_i$ we may further assume that $a_0 = 1$. In this case we have

$$N = \ker(l) = \{(x_0, \dots, x_n);\ x_0 = -\sum_{i=1}^n a_ix_i\}.$$

Recall that the morphism of schemes $f : \mathsf{Spec}(A) \to \mathbb{P}^n_{\mathbb{Z}}$ deduced from N is induced from the graded morphism

$$\mathbb{Z}[t_0, \dots, t_n] \to A[u]$$

that sends t_i to a_iu. As the image of t_0 is invertible, f factors trough $\mathrm{D}_+(t_0)$ and the morphism is given by

$$\mathbb{Z}[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}] \to A$$

where $\frac{t_i}{t_0}$ is sent to a_i. Moreover, $\mathcal{T}_{|\mathrm{D}_+(t_0)}$ correspond to the kernel of the morphism

$$\mathbb{Z}[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}]^{n+1} \to \mathbb{Z}[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}].t_0$$

which sends the vector e_i to t_i. Applying $- \otimes_{\mathbb{Z}[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}]} A$ we see that $f^\star\mathcal{T}(\mathsf{Spec}(A))$ is the kernel of the morphism $A^{n+1} \to A$ that sends e_0 to a_i. This exactly the A-module N. The theorem is proved. ☐

Corollary 2.91 — *Let $c : X \to \mathsf{Spec}(A)$ be a morphism of schemes. There is a canonical bijection from the set of quasi-coherent $\mathcal{O}_X$-submodules $\mathcal{N} \subset \mathcal{O}_X^{n+1}$ such that $\mathcal{O}_X^{n+1}/\mathcal{N}$ is locally free of rank one and the set of morphisms of schemes $f : X \to \mathbb{P}^n_A$ making the triangle*

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{P}^n_A \\ & \searrow^{c} & \downarrow \\ & & \mathsf{Spec}(A) \end{array}$$

commutative. Moreover, this bijection is compatible with the change of rings in the following sense. Let $A \to A'$ be a morphism of rings. Denote $X' = X \times_{\mathsf{Spec}(A)} \mathsf{Spec}(A')$, $p : X' \to X$, $\mathcal{N}' = p^\star \mathcal{N}$ and $f' : X' \to \mathbb{P}^n_A \times_{\mathsf{Spec}(A)} \mathsf{Spec}(A') \simeq \mathbb{P}^n_{A'}$. Then, f' corresponds to $\mathcal{N}' \subset \mathcal{O}_{X'}^{n+1}$.

As an application of Theorem 2.90 we define the *Segre morphism*. Let $m, n \in \mathbb{N}$. Let $t_0, \dots, t_m$ and $t'_0, \dots, t'_n$ be two set of independent variables. We consider $\mathbb{P}^m_A = \mathsf{Proj}(A[\underline{t}])$ and $\mathbb{P}^n_A = \mathsf{Proj}(A[\underline{t}'])$. Let $X = \mathbb{P}^m_A \times_{\mathsf{Spec}(A)} \mathbb{P}^n_A$ and denote $pr_1 : X \to \mathbb{P}^m_A$ and $pr_2 : X \to \mathbb{P}^n_A$ the projections to the first and second factors. We define a locally free $\mathcal{O}_X$-module of rank 1 by the formula

$$\mathcal{O}_X(1) = pr_1^\star \mathcal{O}_{\mathbb{P}^m_A}(1) \otimes_{\mathcal{O}_X} pr_2^\star \mathcal{O}_{\mathbb{P}^n_A}(1).$$

We have obvious global sections $t_i \otimes t'_j \in \mathcal{O}_X(1)(X)$ for $0 \leq i \leq m$ and $0 \leq j \leq n$. Moreover, the induced morphism

$$\mathcal{O}_X^{(m+1)(n+1)} \to \mathcal{O}_X(1)$$

is surjective. By Theorem 2.90 there is an obvious morphism

$$X \to \mathbb{P}^{mn+m+n}_{\mathbb{Z}}.$$

Using the obvious morphism $X \to \mathsf{Spec}(A)$ we get also a morphism

$$X \to \mathbb{P}^{mn+m+n}_A$$

Definition 2.92 — *The morphism*

$$\mathrm{S} : \mathbb{P}^m_A \times_{\mathsf{Spec}(A)} \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$$

we just defined is called the Segre morphism *(relative to A).*

It is obvious from the definition of S that the square

$$\begin{array}{ccc} \mathbb{P}^m_A \times_{\mathsf{Spec}(A)} \mathbb{P}^n_A & \xrightarrow{\mathrm{S}} & \mathbb{P}^{mn+m+n}_A \\ \downarrow & & \downarrow \\ \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} & \xrightarrow{\mathrm{S}} & \mathbb{P}^{mn+m+n}_{\mathbb{Z}} \end{array}$$

is commutative. It follows immediately that this square is even cartesian as the two squares

$$\begin{array}{ccc} \mathbb{P}^m_A \times_{\mathsf{Spec}(A)} \mathbb{P}^n_A & \longrightarrow & \mathsf{Spec}(A) \\ \downarrow & & \downarrow \\ \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} & \longrightarrow & \mathsf{Spec}(\mathbb{Z}) \end{array} \qquad \begin{array}{ccc} \mathbb{P}^{mn+m+n}_A & \longrightarrow & \mathsf{Spec}(A) \\ \downarrow & & \downarrow \\ \mathbb{P}^{mn+m+n}_{\mathbb{Z}} & \longrightarrow & \mathsf{Spec}(\mathbb{Z}) \end{array}$$

are cartesian. This can be used to reduce most statement on Segre morphisms relative to A to Segre morphisms relative to $\mathbb{Z}$. As an application, we have:

Proposition 2.93 — *The Segre morphism* $\mathrm{S} : \mathbb{P}^m_A \times_{\mathsf{Spec}(A)} \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$ *is a closed immersion.*

Proof. We may assume that $A = \mathbb{Z}$. The question is local with respect to the target

$$\mathbb{P}^{mn+m+n}_{\mathbb{Z}} = \mathsf{Proj}(\mathbb{Z}[t_i \otimes t'_j;\ 0 \leq i \leq m \text{ and } 0 \leq j \leq n]).$$

The latter is covered by the $(m+1)(n+1)$ affine opens $\mathrm{D}_+(t_i \otimes t'_j)$. The image of $\mathrm{D}_+(t_i) \times \mathrm{D}_+(t'_j)$ is contained in $\mathrm{D}_+(t_i \otimes t'_j)$. Indeed, the morphism

$$\mathsf{Spec}(\mathbb{Z}[\frac{t_k}{t_i};\ k \neq i] \otimes \mathbb{Z}[\frac{t'_l}{t'_j};\ l \neq j]) \simeq \mathrm{D}_+(t_i) \times \mathrm{D}_+(t'_j) \to \mathbb{P}^{mn+m+n}_{\mathbb{Z}}$$

is induced by the morphism of graded rings

$$\mathbb{Z}[t_i \otimes t'_j;\ 0 \leq i \leq m \text{ and } 0 \leq j \leq n] \to (\mathbb{Z}[\frac{t_k}{t_i};\ k \neq i] \otimes \mathbb{Z}[\frac{t'_l}{t'_j};\ l \neq j])[u]$$

sending $t_k \otimes t'_l$ to $\frac{t_k}{t_i} \otimes \frac{t'_l}{t'_j} u$. The claim is now clear. Moreover, the morphism $\mathrm{D}_+(t_i) \times \mathrm{D}_+(t'_j) \to \mathrm{D}_+(t_i \otimes t'_j)$ is induced by the morphism of rings

$$\mathbb{Z}[\frac{t_k \otimes t'_l}{t_i \otimes t'_j};\ (k,l) \neq (i,j)] \to \mathbb{Z}[\frac{t_k}{t_i},\ \frac{t'_l}{t'_j}; k \neq i \text{ and } l \neq j]$$

which is obviously surjective. In particular, $\mathrm{D}_+(t_i) \times \mathrm{D}_+(t'_j) \to \mathrm{D}_+(t_i \otimes t'_j)$ is a closed immersion.

Now, we have a commutative square

$$\begin{array}{ccc} \mathrm{D}_+(t_i) \times \mathrm{D}_+(t'_j) & \xrightarrow{\mathrm{S}_i} & \mathrm{D}_+(t_i \otimes t'_j) \\ \downarrow & & \downarrow \\ \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} & \xrightarrow{\mathrm{S}} & \mathbb{P}^{mn+m+n}_{\mathbb{Z}}. \end{array}$$

To end the proof we may show that this square is cartesian, i.e., that

$$u : \mathrm{D}_+(t_i) \times \mathrm{D}_+(t'_j) \to \mathrm{S}^{-1}(\mathrm{D}_+(t_i \otimes t'_j))$$

is invertible. This is clearly an open immersion (more precisely, the inclusion of two open subschemes of $\mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}$. It remains to show that u is a closed immersion. Now, $\mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}$ is a separated scheme. It follows from Corollary 2.66 that S is a separated morphism. It is then also the case for the morphism $\mathrm{S}'_i : \mathrm{S}^{-1}(\mathrm{D}_+(t_i \otimes t'_j)) \to \mathrm{D}_+(t_i \otimes t'_j)$. Now $\mathrm{S}_i = \mathrm{S}'_i \circ u$ is a closed immersion. It follows from Proposition 2.65 that u is a closed immersion. This ends the proof of the proposition. □

Let X be a scheme and k be a field. A k-valued point of X is by definition a morphism $\mathsf{Spec}(k) \to X$. We put $X(k) = \hom(\mathsf{Spec}(k), X)$. The following proposition is a special case of the above discussion.

Proposition 2.94 — *Let k be a field. There is a canonical bijection $\mathbb{P}^n_{\mathbb{Z}}(k)$ with the set $(k^{n+1})^{\vee} - \{0\}/k^{\times}$ where $(k^{n+1})^{\vee}$ is the dual of the k-vector space k^{n+1} and $k^{\times}$ is the group of non-zero elements of k acting by multiplication. Taking the dual basis $(e_0^{\vee}, \ldots, e_n^{\vee})$ of the canonical basis, yields a bijection between $\mathbb{P}^n_{\mathbb{Z}}$ and the* *set of equivalent classes* $[a_0 : \cdots : a_n]$ *of* $n+1$*-tuples* $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$ *up to multiplication by an element of* $k^\times$.

Morover, the Segre morphism $\mathrm{S} : \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^{mn+m+n}_{\mathbb{Z}}$ *takes a couple of* k*-valued points* $([a_0 : \cdots : a_m], [b_0 : \cdots : b_n])$ *to the* k*-valued point* $[a_0 b_0 : \cdots : a_m b_n]$ *(an order for the set* $[\![0, m]\!] \times [\![0, n]\!]$ *being fixed).*

3. Properties of Schemes and Morphisms of Schemes

3.1. Relative schemes and base change.

Let A be a ring. Recall that an A-algebra B is an A-module B together with a morphism of A-modules $B \otimes_A B \to B$ making B into a unital commutative ring. This is equivalent to give an ring B and a morphism of rings $A \to B$. A morphism of A-algebra $B \to C$ is a morphism of rings which is A-linear. This is equivalent to ask for the commutativity of the triangle

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & \nearrow & \\ A. & & \end{array}$$

Definition 3.1 — *An A-ringed space $X = (|X|, \mathcal{O}_X)$ is a couple consisting of a topological space $|X|$ and a sheaf of A-algebras $\mathcal{O}_X$ on $|X|$. We say that X is a locally A-ringed space if for every $x \in X$, $\mathcal{O}_{X,x}$ is a local A-algebra (i.e., a local ring).*

A morphism of (locally) A-ringed spaces $Y = (|Y|, \mathcal{O}_Y) \to X = (|X|, \mathcal{O}_X)$ is a morphism of (locally) ringed spaces such that $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is a morphism of sheaves of A-algebras.

Lemma 3.2 — *Let $X = (|X|, \mathcal{O}_X)$ be a ringed space. To make X into an A-ringed space is equivalent to make $\mathcal{O}_X(|X|)$ into an A-algebra. Moreover, if X is a locally ringed space, this is equivalent to give a morphism of locally ringed spaces $X \to \mathsf{Spec}(A)$.*

Proof. Indeed, assume that $\mathcal{O}_X(|X|)$ is an A-algebra. If $|U| \subset |X|$ is open, we have a structure of an A-algebra on $\mathcal{O}_X(|U|)$ given by the composition

$$A \to \mathcal{O}_X(|X|) \to \mathcal{O}_{|X|}(|U|).$$

It is clear that $\mathcal{O}_X$ becomes a sheaf of A-algebras in this way. On the other hand, if X is an A-ringed space, $\mathcal{O}_X(|X|)$ is an A-algebra by definition.

Finally, if X is locally ringed, a morphism $A \to \mathcal{O}_X(|X|)$ is equivalent to a morphism of locally ringed spaces $X \to \mathsf{Spec}(A)$. □

Definition 3.3 — *Let A be a ring. An A-scheme is one of the following equivalent datum:*

1. *A scheme X together with a structure of an A-ringed space.*
2. *A scheme X together with a structure of an A-algebra on $\mathcal{O}_X(|X|)$.*
3. *A scheme X together with a morphism of rings $A \to \mathcal{O}_X(|X|)$.*
4. *A morphism of schemes $X \to \mathsf{Spec}(A)$.*

Definition 3.4 — *Let S be a scheme. An S-scheme is a couple (X, e) consisting of a scheme X and a morphism $e : X \to S$. We usually abuse notation and say simply that X is an S-scheme. The scheme S is often called the base scheme.*

Definition 3.5 — *Let $g : S' \to S$ be a morphism of schemes. Given an S-scheme X, the S'-scheme $X \times_S S'$ is called the* base-change *of X by g.*

3.2. **S-schemes associated to quasi-coherent $\mathcal{O}_S$-algebras.**

Let S be a scheme. A quasi-coherent $\mathcal{O}_S$-algebra is a quasi-coherent $\mathcal{O}_S$-module $\mathcal{A}$ endowed with a morphism of $\mathcal{O}_S$-modules $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$ that makes $\mathcal{A}(U)$ into an $\mathcal{O}_S(U)$-algebra for every open subscheme $U \subset S$.

We may define an S-scheme $\mathsf{Spec}(\mathcal{A})$ as follows. Consider pairs $(U, \mathfrak{p})$ where $U \subset S$ is an affine open subscheme and $\mathfrak{p} \subset \mathcal{A}(U)$ is a prime ideal. Given two such pairs $(U, \mathfrak{p})$ and $(V, \mathfrak{q})$ we write $(U, \mathfrak{p}) \sim (V, \mathfrak{q})$ if there exists $Q \subset U \cap V$ an affine open subscheme and a prime ideal $\mathfrak{r} \subset \mathcal{A}(Q)$ such that $\mathfrak{p} = (\rho^U_Q)^{-1}(\mathfrak{r})$ and $\mathfrak{q} = (\rho^V_Q)^{-1}(\mathfrak{r})$. We define in this way an equivalence relation. Only transitivity of $\sim$ needs a proof. Let $(W, \mathfrak{s})$ be a third pair, and assume there is $R \subset V \cap W$ affine and $\mathfrak{t} \subset \mathcal{A}(R)$ prime such that $\mathfrak{q} = (\rho^V_R)^{-1}(\mathfrak{t})$ and $\mathfrak{s} = (\rho^W_R)^{-1}(\mathfrak{t})$.

Now, let $\mathfrak{r}_0 = (\mathcal{O}_S(Q) \to \mathcal{A}(Q))^{-1}(\mathfrak{r})$. As V is affine, we may find $f \in \mathcal{O}_S(V)$ such that $\mathrm{D}(f)$ is a neighborhood of $\mathfrak{r}_0$ contained in $Q \simeq \mathsf{Spec}(\mathcal{O}_S(Q))$. As $\mathcal{A}$ is quasi-coherent, the canonical morphism $\mathcal{A}(V)_f \to \mathcal{A}(\mathrm{D}(f))$ is invertible. Denote $\mathfrak{r}'$ the image of $\mathfrak{r}_f$ by this isomorphism. As $f \notin \mathfrak{r}_0$, we conclude that $\mathfrak{r}'$ is a prime ideal and $\mathfrak{r} = (\rho^V_{\mathrm{D}(f)})^{-1}(\mathfrak{r}')$. Replacing Q by $\mathrm{D}(f)$ and $\mathfrak{r}$ by $\mathfrak{r}'$, we see that we may assume that $Q = \mathrm{D}(f)$. Similarly, we may assume $R = \mathrm{D}(g)$ for some $g \in \mathcal{O}_S(V)$.

Now we have

$$(\mathcal{O}_S(V) \to \mathcal{A}(Q))^{-1}(\mathfrak{r}) = (\mathcal{O}_S(V) \to \mathcal{Q}(V))^{-1}(\mathfrak{q}) = (\mathcal{O}_S(V) \to \mathcal{A}(R))^{-1}(\mathfrak{s})$$

we see that we may replace Q and R by $\mathrm{D}(fg)$. In particular, we may assume that $Q = R$. But then, $\mathfrak{r} = \mathfrak{t}$ as they both restrict to the same ideal $\mathfrak{q}$ is $\mathcal{A}(V)$. (Use the fact that $\mathcal{A}$ is quasi-coherent to see that $\mathsf{Spec}(\mathcal{A}(Q)) \to \mathsf{Spec}(\mathcal{A}(V))$ is an open immersion.)

We now define $|\mathsf{Spec}(\mathcal{A})|$ to be the set of equivalence classes of pairs $(U, \mathfrak{p})$. There is an obvious morphism $\pi : |\mathsf{Spec}(\mathcal{A})| \to |S|$ which sends the equivalence class of $(U, \mathfrak{p})$ to the image of $\mathfrak{p}$ by $\mathsf{Spec}(\mathcal{A}(U)) \to \mathsf{Spec}(\mathcal{O}_S(U)) \simeq U$.

Lemma 3.6 — *Let $U \subset S$ be an affine open subscheme. The canonical morphism $\alpha_U : |\mathsf{Spec}(\mathcal{A}(U))| \to \mathsf{Spec}(\mathcal{A})$ that sends a prime $\mathfrak{p} \subset \mathcal{A}(U)$ to the equivalence class of the pair $(U, \mathfrak{p})$ defines a bijection from $|\mathsf{Spec}(\mathcal{A}(U))|$ to $\pi^{-1}(|U|)$.*

Proof. It is easy to see that our map is injective and its image lies in $\pi^{-1}(U)$. We only need to check surjectivity. The class of $(V, \mathfrak{q})$ is in $\pi^{-1}(|U|)$ if and only if, the image of $\mathfrak{q}$ by $\mathsf{Spec}(\mathcal{A}(V)) \to V$ is in $V \cap U$. Let $W \subset U \cap V$ be an affine neighborhood of $\pi(\mathfrak{q})$. We may assume that $W = \mathrm{D}(a)$ for some $a \in \mathcal{O}_S(V)$. As $\mathcal{A}$ is quasi-coherent, we know that $\mathcal{A}(W) \simeq \mathcal{A}(V)_a$. As $a \notin \pi(\mathfrak{q})$, the ideal $\mathfrak{q}_a \subset \mathcal{A}(V)_a$ is prime. Thus, we may replace $(V, \mathfrak{q})$ by $(W, \mathfrak{q}_a)$ and also by $(U, (\mathcal{A}(U) \to \mathcal{A}(W))^{-1}\mathfrak{q}_a)$. □

We endow $|\mathsf{Spec}(\mathcal{A})|$ with the topology generated by $\alpha_U(\mathrm{D}(f))$ for $U \subset S$ affine and $f \in \mathcal{A}(U)$.

We now define the structure sheaf for the scheme $\mathsf{Spec}(\mathcal{A})$. For simplicity, we will assume that S is separated. We take the kernel of the following morphism of sheaves

$$\prod_{U \subset S} \alpha_{U*}\mathcal{O}_{\mathsf{Spec}(\mathcal{A}(U))} \to \prod_{U \subset S, V \subset S} \alpha_{U \cap V*}\mathcal{O}_{\mathsf{Spec}(\mathcal{A}(U \cap V))}$$

where U and V run among affine open subschemes of S. Given an open set $W \subset |\mathsf{Spec}(\mathcal{A})|$ the above arrow sends a family of sections $(a_U \in \mathcal{O}(\alpha_U^{-1}(W)))_U$ to the family $((a_U)_{|\alpha_{U\cap V}^{-1}(W)} - (a_V)_{|\alpha_{U\cap V}^{-1}(W)})_{(U,V)}$. It is easily checked that $\mathsf{Spec}(\mathcal{A}) = (|\mathsf{Spec}(\mathcal{A}), \mathcal{O})$ is a ringed space which is locally isomorphic to $\mathsf{Spec}(\mathcal{A}(U))$. This implies that $\mathsf{Spec}(\mathcal{A})$ is a scheme. Furthermore, we have an obvious morphisms of locally ringed spaces $\mathsf{Spec}(\mathcal{A}) \to S$, giving $\mathsf{Spec}(\mathcal{A})$ the structure of an S-scheme.

Definition 3.7 — *A morphism of schemes $p : X \to S$ is called* affine *if there exists an open covering $(U_i)_{i\in I}$ of S by affine subscheme such that $f^{-1}(U_i)$ are affine for all $i \in I$. An S-scheme is affine if its structure morphism is affine.*

Proposition 3.8 — *Let S be a scheme and $p : X \to S$ an affine S-scheme. Let $\mathcal{A} = p_*\mathcal{O}_X$. Then $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra and there is a canonical isomorphism of S-schemes $X \simeq \mathsf{Spec}(\mathcal{A})$. In particular, for every affine open subset $U \subset S$, $p^{-1}(U)$ is affine.*

Proof. To show that $\mathcal{A}$ is quasi-coherent, it suffices to show that for every $i \in I$ and $f \in \mathcal{O}_S(U_i)$ the obvious morphism $\mathcal{A}(U_i)_f \to \mathcal{A}(\mathrm{D}(f))$ is invertible. In other words, we need to show that

$$\mathcal{O}_X(p^{-1}(U_i))_{f\circ p} \to \mathcal{O}_X(\mathrm{D}(f \circ p))$$

is invertible. This follows from the hypothesis that $p^{-1}(U_i)$ is affine.

We define a morphism $X \to \mathsf{Spec}(A)$ as follows. Set $X_i = p^{-1}(U_i)$. The family $(X_i)_{i\in I}$ is a open covering of X by affine subschemes. To define a morphism $f : X \to \mathsf{Spec}(\mathcal{A})$ we need to construct $f_i : X_i \to \mathsf{Spec}(\mathcal{A})$ such that f_i and f_j have the same restrictions to $X_i \cap X_j$.

We take for f_i the isomorphism $X_i \xrightarrow{\sim} \mathsf{Spec}(\mathcal{O}_X(X_i)) = \mathsf{Spec}(\mathcal{A}(U_i))$. To see that $f_{i|X_i\cap X_j} = f_{j|X_i\cap X_j}$ we choose an affine open subscheme $V \subset U_i \cap U_j$ and check that $f_{i|p^{-1}V} = f_{j|p^{-1}(V)}$. But, $p^{-1}(V)$ is canonically isomorphic to $\mathsf{Spec}(\mathcal{A}(V))$ and both restrictions coïcides with the obvious morphism. □

3.3. **S-schemes associated to quasi-coherent graded $\mathcal{O}_S$-algebras.**

Now, let $\mathcal{A} = \oplus_{n\in\mathbb{N}}\mathcal{A}_n$ be an $\mathbb{N}$-graded $\mathcal{O}_S$-algebra. We want to define an S-scheme $\mathsf{Proj}(\mathcal{A})$. The set $|\mathsf{Proj}(\mathcal{A})|$ will be the set of equivalence classes of pairs $(U, \mathfrak{p})$ where $U \subset S$ is an affine open subscheme and $\mathfrak{p} \subset \mathcal{A}(U)$ is a homogenous prime ideal not containing the ideal $\mathcal{A}_+(U)$. Note that we may view $|\mathsf{Proj}(\mathcal{A})|$ as a subset of $|\mathsf{Spec}(\mathcal{A})$. In particular, it has a natural topology. Let $\pi : |\mathsf{Proj}(\mathcal{A})| \to |S|$ be the obvious map. We get from Lemma 3.6:

Lemma 3.9 — *Let $U \subset S$ be an affine open subset. The obvious morphism $\alpha'_U : |\mathsf{Proj}(\mathcal{A}(U))| \to \pi^{-1}(U)$ is a homeomorphism to an open subset of $|\mathsf{Proj}(\mathcal{A})|$.*

To describe the scheme structure on $|\mathsf{Proj}(\mathcal{A})|$, we assume for simplicity that S is separated and take the kernel of

$$\prod_{U\subset S} \alpha'_{U*}\mathcal{O}_{\mathsf{Proj}(\mathcal{A}(U))} \to \prod_{U\subset S, V\subset S} \alpha'_{U\cap V*}\mathcal{O}_{\mathsf{Proj}(\mathcal{A}(U\cap V))}.$$

By construction, $\mathsf{Proj}(\mathcal{A})$ is a scheme which is naturally covered by $\mathsf{Proj}(\mathcal{A}(U))$ for $U \subset S$ affine.

Definition 3.10 — *Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ a quasi-coherent sheaf of ideals. The blow-up of $\mathcal{I}$ is the scheme $\mathrm{Bl}_{\mathcal{I}}(X) = \mathsf{Proj}(\oplus_{n=0}^{\infty}\mathcal{I}^n)$ endowed the canonical morphism $e : \mathrm{Bl}_{\mathcal{I}}(X) \to X$.*

Proposition 3.11 — *The ideal $(e^*\mathcal{I})\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)}$ is locally principal. Conversely, let $f : Y \to X$ be a morphism of schemes such that $(f^*\mathcal{I})\mathcal{O}_Y$ is a locally principal ideal. Then there exists a unique morphism $Y \to \mathrm{Bl}_{\mathcal{I}}(X)$ making the triangle*

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Bl}_{\mathcal{I}}(X) \\ & \searrow & \downarrow \\ & & X \end{array}$$

commutative.

Proof. To see that $(e^*\mathcal{I})\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)}$ is locally principle, we may assume that $X = \mathsf{Spec}(A)$ is affine and $\mathcal{I} = \tilde{I}$ for $I \subset A$ an ideal. Then $\mathrm{Bl}_{\mathcal{I}}(X) = \mathsf{Proj}(\oplus_{n\in\mathbb{N}} I^n)$.

Let $G = \oplus_{n\in\mathbb{N}} I^n$. As G_+ is generated by $G_1 = I$, we may cover $\mathsf{Proj}(G)$ by $\mathsf{Spec}((G_f)_0)$ with $f \in G_1$. It remains to show that the ideal $I.(G_f)_0$ is principal. But, the latter is clearly generated by f as $g = f.\left(\dfrac{g}{f}\right)$ for all $g \in I$.

To establish the universal property, we may assume that $X = \mathsf{Spec}(A)$ is affine, $Y = \mathsf{Spec}(B)$ is affine and IB is principal. Let $r \in IB$ be a generator. For $f \in I$, we may write $f = a(f).r$ with $a(f) \in B$. As IB is generated by $f \in I$, $\mathsf{Spec}(B) = \bigcup_{f\in I} \mathrm{D}(a(f))$. As the problem is local on Y, we may assume that IB is generated by $f \in I$. Thus, for every $g \in I$, we may find $a(g) \in B$ such that $a(g)f = g$. We define the morphism $\mathsf{Spec}(B) \to \mathsf{Proj}(G)$ to be the one that factors trough $\mathrm{D}_+(f)$ and is given on functions by

$$A\left[\frac{I}{f}\right] \to B$$

sending $\dfrac{g}{f}$ to $a(g)$. We leave it as an exercise to check the details. □

3.4. Morphisms of finite type and of finite presentation.

Let A be a ring. An A-algebra B is finitely generated if there exists finitely many elements $b_1, \ldots, b_n \in B$ such that the morphism

$$A[t_1, \ldots, t_n] \to B, \tag{10}$$

which sends t_i to b_i, is surjective. If moreover, we may chose $b_1, \ldots, b_n$ so that the kernel of (10) is a finitely generated ideal of $A[t_1, \ldots, t_n]$, we say that B is a finitely presented A-algebra.

Lemma 3.12 — *Let B be a finitely presented A-algebra. There exists a subring $A_0 \subset A$ finitely generated over $\mathbb{Z}$ and a finitely generated A_0-algebra B_0 such that $B \simeq A \otimes_{A_0} B_0$.*

Proof. Indeed, let $(P_1, \ldots, P_n)$ be the kernel of (10). We may take A_0 the subring of A generated by the coefficients of the polynomials P_i and B_0 the A_0-algebra $A_0[t_1, \ldots, t_n]/(P_1, \ldots, P_n)$. □

Corollary 3.13 — *Let $e : B \to C$ be a morphism of A-algebras. Assume that B is finitely generated, C is finitely presented and e is surjective. Then* $\ker(e)$ *is a finitely generated ideal of B.*

Proof. We may assume that $B = A[t_1, \dots, t_m]$. Using Lemma 3.12, we may assume that $e = A \otimes_{A_0} e_0$ for some morphism $e_0 : A_0[t_1, \dots, t_m] \to C_0$ which we may also assume to be surjective. Then $\ker(e) = \ker(e_0)A[t_1, \dots, t_n]$. As $A_0[t_1, \dots, t_n]$ is finitely generated over $\mathbb{Z}$, it is noetherian. In particular, $\ker(e_0)$ is finitely generated. □

Lemma 3.14 — *Let A be a ring, B an A-algebra and C a B-algebra. Assume that B is finitely generated (resp. presented) over A and C is finitely generated (resp. presented) over B. Then C is finitely generated (resp. presented) over A.*

Definition 3.15 — *Let $f : X \to S$ be a morphism of schemes. We say that f is* locally of finite type *(resp.* locally of finite presentation*) if for any $x \in X$, there exist affine open neighborhoods $x \in U \subset X$ and $f(x) \in T \subset S$ such that $f(X) \subset U$ and $\mathcal{O}_X(U)$ is a finitely generated (resp. finitely presented) $\mathcal{O}_S(T)$-algebra.*

Lemma 3.16 — *Let $f : X \to S$ be a morphism of affine schemes. Then f is locally of finite type (resp. locally of finite presentation) if and only if $\mathcal{O}_X(|X|)$ is a finitely generated (resp. finitely presented) $\mathcal{O}_S(|S|)$-algebra.*

Proof. We first deal with finite generation. We may assume that $X = \mathsf{Spec}(B)$ and $S = \mathsf{Spec}(A)$. There exists $f_1, \dots, f_n \in B$ such that B_{f_i} are finitely generated A-algebras and $\sum_{i=1}^n f_i = 1$. Thus, we may find $x_{i1}, \dots, x_{in_i}$ such that $\frac{x_{ij}}{f_i^{e_j}}$ generate the A-algebra B_{f_i}. But then, the family of (x_{ij}, f_i) generates the ring B. Indeed, if B' designate the subring of B generated by these elements, then $B'_{f_i} = B_{f_i}$. As $\sum_{i=1}^n f_i = 1$ holds also in B', we see that the morphism of B'-modules $B' \to B$ is necessary invertible.

Assume now that the B_{f_i} are finitely presented. We know that B is finitely generated. Let f_i be as before. Choose a surjective morphism

$$e : A[t_1, \dots, t_m, s_1, \dots, s_{n-1}] \to B$$

sending s_i to f_i. We set also $s_n = 1 - \sum_{i=1}^{n-1} s_i$. We have surjective morphisms (for $1 \leq i \leq n$)

$$e_{s_i} : A[t_1, \dots, t_m, s_1, \dots, s_{n-1}]_{s_i} \to B_{f_i}.$$

As B_{f_i} is finitely presented over A, the kernel of e_{s_i} is finitely generated. Now $\ker(e_{s_i}) = \ker(e)_{s_i}$. As $\sum_{i=1}^n s_i = 1$, we obtain that $\ker(e)$ is finitely generated. □

Definition 3.17 — *A scheme X is* quasi-compact *if it can be covered by finitely may affine open subschemes. Equivalently, X is quasi-compact if from every open cover of X we can extract a finite open cover.*

A morphism of schemes $f : X \to S$ is quasi-compact if for every affine subscheme $T \subset S$, $f^{-1}(T)$ is quasi-compact.

Definition 3.18 — *A morphism $f : X \to S$ is of finite type (resp. of finite presentation) if it is locally so and is also quasi-compact.*

The following lemma is an easy exercise.

Lemma 3.19 — *Let $f : X \to S$ be an S-scheme and $f' : X' \to S'$ denote the base change of f along a morphism $S' \to S$. If f is (locally) of finite type (resp. of finite presentation) the so is f'.*

3.5. Noetherian and locally noetherian schemes.

Recall that a ring A is noetherian if every ideal of A is finitely generated. This equivalent to the condition that every increasing chain of ideals is stationary.

Definition 3.20 — *Let X be a scheme. We say that X is* locally noetherian *if every point $x \in X$ admits an affine neighborhood U such that $\mathcal{O}_X(U)$ is a noetherian ring. We say that X is* noetherian *if it is locally noetherian and quasi-compact (i.e., can be covered by a finite number of affine opens).*

Lemma 3.21 — *Let S be a (locally) noetherian scheme and X be a (locally) of finite type S-scheme. Then X is (locally) noetherian and (locally) of finite presentation over S.*

Proof. The question is local on S and X. So we may assume that $S = \mathsf{Spec}(A)$ and $X = \mathsf{Spec}(B)$. By hypothesis, A is a noetherian ring and $B$is a finitely generated A-algebra. It is well-known that this implies that B is noetherian and finitely presented over A. □

Let A be a ring. A prime ideal $\mathfrak{p} \subset A$ is minimal if no prime ideal is strictly contained in $\mathfrak{p}$. We will use the following well-known fact about noetherian rings:

Lemma 3.22 — *Let A be a noetherian ring. Then A has finitely many minimal prime ideals.*

Proof. We may assume that A is reduced. We argue by contradiction. Let $(\mathfrak{p}_i)_{i\in\mathbb{N}}$ be a family of mutually distinct minimal prime ideals of A. The localized ring $A_{\mathfrak{p}_i}$ is artinian and reduced, which implies that $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$. As $\mathfrak{p}_i$ is finitely generated, we may find $f_i \notin \mathfrak{p}_i$ such that $f_i\mathfrak{p}_i = 0$.

For $j \neq i$, we assumed that $\mathfrak{p}_i \neq \mathfrak{p}_j$. As $0 = f_i\mathfrak{p}_i \subset \mathfrak{p}_j$, we necessarily have $f_i \in \mathfrak{p}_j$.

To conclude, we consider the ideals $\mathfrak{a}_n = \cap_{i\geq n}\mathfrak{p}_i$. We clearly have $\mathfrak{a}_n \subset \mathfrak{a}_{n+1}$, $f_n \in \mathfrak{a}_{n+1}\backslash\mathfrak{a}_n$. This contradicts the assumption that A is noetherian. □

Definition 3.23 — *Let X be a topological space. We say that X is* irreducible *if X is non-empty and is not the union of two closed and strict subsets of X. Otherwise, we say that X is* reducible.

Let X be a topological space. Then X is irreducible if and only if X is non-empty and any non-empty open subset of X is dense.

Let $Y \subset X$ be a subset of X (endowed with the induced topology). Then Y is irreducible if and only if its closure $\overline{Y}$ is irreducible. It follows that any irreducible subset of X is contained in at least one maximal irreducible subset of X which is necessarily closed. The maximal irreducible subset of X are called the *irreducible components* of X. It is clear that X is the set-theoretic union of its irreducible components.

Lemma 3.24 — *Let $X = \mathsf{Spec}(A)$ be an affine scheme. A closed subset $Y \subset X$ is irreducible if and only if it is the zero set $\mathcal{Z}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subset A$. In particular, the irreducible components of X corresponds to the minimal prime ideals of A*

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal. Then $\mathcal{Z}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is the closure of a point. Thus, it is an irreducible closed subset.

Conversely, let $Y \subset X$ be an irreducible closed subset. We may write $Y = \mathcal{Z}(\mathfrak{a})$ with $\mathfrak{a} \subset A$ an ideal such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$. We have

$$\mathfrak{a} = \bigcap_{\mathfrak{a}\subset\mathfrak{p},\, \mathfrak{p}/\mathfrak{a} \text{ minimal}} \mathfrak{p}.$$

We need to show that there is exactly one prime $\mathfrak{p}$ containing $\mathfrak{a}$ and which is minimal for this property. We argue by contradiction, and we fix two distinct such primes $\mathfrak{p}$ and $\mathfrak{q}$. As $(A/\mathfrak{a})_{\mathfrak{p}}$ is a field, for each $g \in \mathfrak{p}$ there exists $f \in A \backslash \mathfrak{p}$ such that $fg \subset \mathfrak{a}$. Taking $g \in \mathfrak{p} \backslash \mathfrak{q}$, we get $f \in \mathfrak{q}$. This means that $\mathcal{Z}(\mathfrak{a}) \subset \mathcal{Z}(\mathfrak{p}) \cup \mathcal{Z}(f)$ with $\mathfrak{p} \notin \mathcal{Z}(f)$ and $\mathfrak{q} \in \mathcal{Z}(f)$. We have reached a contradiction. □

Corollary 3.25 — *Let X be a noetherian scheme. Then $|X|$ has finitely many irreducible components.*

Proof. As X is quasi-compact, we may write $X = \cup_{i\in I} U_i$ where U_i are affine open subschemes. If $Z \subset X$ is an irreducible component of X, the $Z \cap U_i$ is empty or an irreducible component of U_i (as $Z \cap U_i$ is empty or dense in Z). Thus, it is sufficient to prove the lemma for each U_i. In particular, we may assume $X = \mathsf{Spec}(A)$ affine. This case follows from Lemmas 3.22 and 3.24. □

Lemma 3.26 — *Let X be an irreducible scheme. Then there exist an unique point $\eta_X \in |X|$ such that $|X| = \overline{\{\eta_X\}}$. This is the* generic point *of X.*

Proof. Let $U \subset X$ be an non-empty affine open subscheme of X. Then U is dense in X and is irreducible. As $U \simeq \mathsf{Spec}(\mathcal{O}_X(U))$, the ring $\mathcal{O}_X(U)$ has a unique minimal ideal $\mathfrak{m} \subset \mathcal{O}_X(U)$ which correspond to a point $\eta \in U$. Then $\{\eta\}$ is dense in $|U|$ and hence also in $|X|$. This proves the lemma. □

Definition 3.27 — *Let X be a scheme. A point $\eta \in X$ is called a* generic *point if its closure $\overline{\{\eta\}}$ is an irreducible component of $|X|$.*

3.6. Reduced and integral schemes.

Recall that a ring A is reduced if its nilradical is zero, i.e., it has no non-zero nilpotent elements.

Definition 3.28 — *A scheme X is* reduced *if every $x \in |X|$ admits an affine open neighborhood U such that $\mathcal{O}_X(U)$ is a reduced ring.*

Lemma 3.29 — ***1-*** *Let X be a scheme. Then X is reduced if and only if for every $x \in |X|$, the local ring $\mathcal{O}_{X,x}$ is reduced.*

2- *Let $X = \mathsf{Spec}(A)$ be an affine scheme. Then X is reduced if and only if A is reduced.*

Proof. If X is a reduced scheme, then $\mathcal{O}_{X,x}$ are clearly reduced because a ring of fractions of a reduced ring is again reduced.

Now assume that the local rings of X is reduced and let's show that X is reduced. We may assume that $X = \mathsf{Spec}(A)$ is affine and we will show that A is a reduced ring (proving also the second part of the lemma). But if $\mathfrak{n} \subset A$ denotes the nilradical of A, $\mathfrak{n}_\mathfrak{p} \subset A_\mathfrak{p}$ is the nilradical of $A_\mathfrak{p}$. Thus our assumption implies that $\mathfrak{n}_\mathfrak{p} = 0$ for all prime ideal of A. This show that $\mathfrak{n} = 0$. The lemma is proven. □

Remark 3.30 — Given a scheme X, we denote by $\mathcal{N} \subset \mathcal{O}_X$ the sheaf of ideals such that $\mathcal{N}(U)$ is the nilradical of $\mathcal{O}_X(U)$ for each affine open subscheme $U \subset X$. Then $\mathcal{N}$ is quasi-coherent and $\mathcal{O}_X/\mathcal{N}$ is a quasi-coherent $\mathcal{O}_X$-algebra such that $X_{red} = \mathsf{Spec}(\mathcal{O}_X/\mathcal{N})$ is the biggest closed subscheme of X which is reduced. We call X_{red} the *reduction* of X. The obvious closed imbedding $X_{red} \to X$ is a homeomorphism on the underlying topological spaces.

Definition 3.31 — *A scheme X is* integral *if it is reduced and irreducible.*

Recall that a ring is called an integral domain if it has no zero divisors.

Lemma 3.32 — ***1-*** *Let X be an integral scheme. Then any non-empty open subscheme of X is integral.*

2- *Let $X = \mathsf{Spec}(A)$ be an affine scheme. Then X is integral if and only if A is an integral domain.*

Proof. The first claim is obvious. For the second claim, the condition is sufficient. Assume that X is integral. Then A has a unique minimal prime ideal. This minimal prime ideal is necessary (0) as A is reduced. But if (0) is prime then A is an integral domain. □

Proposition 3.33 — *Let X be a scheme. Then the following conditions are equivalent*

(1) *X is integral,*
(2) *X is connected and for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is an integral domain.*

Proof. The implication (1) ⇒ (2) is obvious. Let's prove the converse. We know that X is reduced. We need to show that X is irreducible.

Assume to the contrary that we can write $X = X_1 \cap X_2$ with $X_i \subset X$ a strict closed subset. We have $X_1 \cap X_2 \neq \emptyset$ because otherwise X will not be connected. Let $x \in X_1 \cap X_2$. The scheme $\mathsf{Spec}(\mathcal{O}_{X,x})$ is not irreducible. Indeed, let $f : \mathsf{Spec}(\mathcal{O}_{X,x}) \to X$ be the obvious inclusion. Then $\mathsf{Spec}(\mathcal{O}_{X,x}) = f^{-1}(X_1) \cup f^{-1}(X_2)$. It remains to show that $f^{-1}(X_i) \neq \mathsf{Spec}(\mathcal{O}_{X,x})$. But if η_i is the generic point of an irreducible component of X_i containing x, we have $\eta_1 \in f^{-1}(X_1)\backslash f^{-1}(X_2)$ and $\eta_2 \in f^{-1}(X_2)\backslash f^{-1}(X_1)$.

Now the affine scheme $\mathsf{Spec}(\mathcal{O}_{X,x})$ being reducible, $(0) \subset \mathcal{O}_{X,x}$ is not a prime ideal. We have reached a contradiction. □

3.7. Normal schemes.

Let A be a ring and B and A-algebra. An element $b \in B$ is *algebraic* over A if there exists a unitary polynomial $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in A[t]$ such that $P(b) = 0$. The set of algebraic elements of B is a sub-algebra of B called the *normalization* of A in B.

An integral domain A is said to be normal if A is its own normalization in its field of fractions $A_{(0)} = \mathrm{Frac}(A)$.

Definition 3.34 — *Let X be a scheme. We say that X is* normal *if for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal integral domain.*

The property for a scheme to be normal is clearly a local property. Moreover, we have:

Proposition 3.35 — *Let $X = \mathsf{Spec}(A)$ be a connected affine scheme. The following properties are equivalent:*

(1) *X is normal.*
(2) *The ring A is normal.*

Proof. If A is normal, then so are the local rings $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset A$. This proves that X is normal.

Now assume that X is normal. As it is connected, Proposition 3.33 implies that X is integral. Hence, A is an integral domain. To see that A is normal, let $a \in \mathrm{Frac}(A)$ be algebraic over A. Then, a is also algebraic over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset A$. As $A_{\mathfrak{p}}$ is normal, we get $a \in A_{\mathfrak{p}}$. Our claim now follows from the well known fact: $A = \cap_{\mathfrak{p} \in \mathsf{Spec}(A)} A_{\mathfrak{p}}$. □

Let A be a ring. We denote by $\mathrm{NZD}(A) \subset A$ the multiplicative subset of non-zero divisors in A. We have the following lemma:

Lemma 3.36 — *Assume that A is reduced. Then*

$$\mathrm{NZD}(A) = A \backslash \bigcup_{\mathfrak{m} \in \mathsf{Spec}(A) \text{ minimal}} \mathfrak{m}.$$

Proof. Let a be a zero divisor, i.e., $ab = 0$ for some $b \neq 0$. As A is reduced, the intersection of all minimal prime ideals in A is (0). Hence, there exists a minimal prime ideal $\mathfrak{m} \subset A$ such that $b \notin \mathfrak{m}$. But then $a \in \mathfrak{m}$.

On the other hand, suppose that $a \in \mathfrak{m}$ where $\mathfrak{m} \subset A$ is a minimal prime ideal. As $A_{\mathfrak{m}}$ is a field (because, reduced and artinian) there exists $f \in A \backslash \mathfrak{m}$ such that $fa = 0$. □

For a general ring A, we denote $\mathrm{Frac}(A) = \mathrm{NZD}(A)^{-1}A$. More generally, given a scheme X, we define an $\mathcal{O}_X$-algebra $\mathrm{Frac}(\mathcal{O}_X)$ by sheafifying the presheaf that sends an open subset U to the $\mathcal{O}_X$-algebra $\mathrm{Frac}(\mathcal{O}_X(U))$. We have the following lemma:

Lemma 3.37 — *Let X be a reduced scheme with finitely many irreducible components $(X_i)_{i \in I}$. Then $\mathrm{Frac}(\mathcal{O}_X)$ is a quasi-coherent $\mathcal{O}_X$-algebra which for an open subscheme U is given by $\mathrm{Frac}(\mathcal{O}_X)(U) = \prod_{i \in I(U)} \kappa(\eta_i)$ where η_i is the generic point of X_i and $I(U)$ is the set of indices i such that $U \cap X_i \neq \emptyset$. Moreover, $\mathsf{Spec}(\mathrm{Frac}(\mathcal{O}_X))$ is isomorphic to the disjoint union $\coprod_{i \in I} \mathsf{Spec}(\kappa(\eta_i))$.*

Proof. We may assume that $X = \mathsf{Spec}(A)$ is affine. Let $\mathfrak{m}_i$ denote the minimal prime ideal corresponding to the generic point of X_i. As $\cap_{j \neq i} \mathfrak{m}_j \not\subset \mathfrak{m}_i$, we may find $h_i \in (\cap_{j \neq i} \mathfrak{m}_j) \backslash \mathfrak{m}_i$. Then $\mathrm{D}(h_i)$ is an open neighborhood of η_i contained in X_i.

Let $h = \sum_{i \in I} h_i$. Then $\mathrm{D}(h) \subset \coprod_{i \in I} \mathrm{D}(h_i)$. As $\mathrm{D}(h_i) \cap \mathrm{D}(h_j) = \emptyset$ for $i \neq j$, we deduce that $\mathrm{D}(h) = \coprod \mathrm{D}_i(h)$ where $\mathrm{D}_i(h) = \mathrm{D}(h) \cap \mathrm{D}(h_i)$. Moreover, $h \notin \cup_{i \in I} \mathfrak{m}_i$. This means that h is a non-zero divisor, i.e., $\mathrm{D}_i(h)$ is an open neighborhood of η_i.

As h is a non-zero divisor, we may replace X by $\mathsf{Spec}(A_f)$. We are thus reduced to the case where X is integral, i.e., A is an integral domain. This case is very easy. □

Let X be a scheme and $\mathcal{A}$ a quasi-coherent $\mathcal{O}_X$-algebra. We define a sub-$\mathcal{O}_X$-algebra $\mathcal{A}_0 \subset \mathcal{A}$ such that $\mathcal{A}_0(U)$ is the set of $\mathcal{O}_X(U)$-algebraic elements in $\mathcal{A}(U)$ for any affine open subscheme $U \subset X$. It is clear that $\mathcal{A}_0$ is a quasi-coherent $\mathcal{O}_X$-algebra.

Definition 3.38 — *Let X be a reduced scheme having finitely many irreducible components. We define $\mathcal{O}_X^{nor}$ to be the sub-$\mathcal{O}_X$-algebra of integral elements in* $\mathrm{Frac}(\mathcal{O}_X)$. *The scheme $X^{nor} = \mathsf{Spec}(\mathcal{O}_X^{nor})$ is called the* normalization *of X.*

Lemma 3.39 — *Under the hypothesis of Definition 3.38, the scheme X^{nor} is normal.*

Proof. This is a local statement. So we may assume that $X = \mathsf{Spec}(A)$ is affine. Let $(\mathfrak{m}_i)_{i\in I}$ denote the minimal prime ideals of A. By hypothesis, there are only finitely many of them.

Let $B \subset \prod_{i\in I} \kappa(\mathfrak{m}_i)$ be the sub-A-algebra of integral elements. For $j \in I$, we denote $e^j = (e^j_i)_{i\in I} \in \prod_{i\in I} \kappa(\mathfrak{m}_i)$ the element given by $e^j_i = 0$ for $j \neq i$ and $e^j_j = 1$. As $(e^j)^2 = e^j$, we have $e^j \in B$. Let $B^j \subset B$ be the ideal of elements $a = (a_i)_{i\in I}$ where $a_i = 0$ unless $i = j$. We clearly have $B^j = B.e^j$. Moreover, B^j is naturally isomorphic to the sub-A-algebra of algebraic elements in $\kappa(\mathfrak{m}_j)$. In particular B^j is a normal ring and $\mathsf{Spec}(B) = \coprod_{j\in I} \mathsf{Spec}(B^j)$. This proves the lemma. □

3.8. Finite and entire morphisms.

Let A be a ring and B an A-algebra. We say that B is an *entire* algebra if every element of B is algebraic over A. When B is finitely generated we say that B is a *finite* A-algebra.

Lemma 3.40 — *Let B be an A-algebra. If B is finite the A-module B is finitely generated. When A is noetherian the converse also holds, i.e., B is finite if and only if the A-module B is finitely generated.*

Proof. Suppose that B is finite. Then there are algebraic elements $b_1, \ldots, b_n \in B$ generating B. If $P_i \in A[t]$ is a unitary polynomial annihilating b_i, we get a surjective morphism

$$A[t_1]/P_1(t_1) \otimes_A \cdots \otimes_A A[t_n]/P_n(t_n) \to B$$

by sending t_i to b_i. It remains to show that $A[t]/P$ is a free A-module of rank $\deg(P) = n$ for $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Indeed its is freely generated by the classes $\bar{t}^0, \ldots, \bar{t}^{n-1}$.

Now assume that A is noetherian and that B is finitely generated as an A-module. It follows that A is finitely generated as an A-algebra. It remains to show that every element of B is algebraic over A. Let $f \in B$ and consider the submodule $A[f]$ of B generated by f^n for $n \in \mathbb{N}$. This module is finitely generated as B is finitely generated and A is noetherian. It follows that $A[f]$ can be generated (as an A-module) by $1, \ldots, f^N$. In particular, we can write $f^{N+1} = \sum_{i=0}^N a_i f^i$. □

Definition 3.41 — *A morphism $f : Y \to X$ is entire (resp.* finite*) if it is affine and any $x \in X$ admits an affine open neighborhood U such that $\mathcal{O}_Y(f^{-1}(U))$ is an entire (resp. a finite) $\mathcal{O}_X(U)$-algebra.*

Example 3.42 — A closed immersion is a finite morphism. If X is a reduced scheme having finitely many irreducible components, the obvious morphism $X^{nor} \to X$ is an entire morphism.

The property for f to be entire or finite is local on X. We have:

Lemma 3.43 — *Let $f : Y \to X$ be an entire (resp. finite) morphism of schemes. Assume that X is affine. Then Y is affine and $\mathcal{O}_Y(Y)$ is an entire (resp. a finite) $\mathcal{O}_X(X)$-algebra.*

Proof. As f is an affine morphism, Y is affine. Assume that $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(B)$. There exists $f_1, \ldots, f_n \in A$ such that $(f_1, \ldots, f_n) = A$ and B_{f_i} is an entire (resp. finite) A_{f_i}-algebra for all i.

First we show that B is entire. Let $b \in B$. We can find unitary polynomials $P_i \in A_{f_i}[t]$ such that $P_i(\frac{b}{1}) = 0$. It is easy to deduce from this the existence of polynomials $Q_i \in A[t]$ of degree d (independent of i) with leading coefficient f_i^r (r being independent of i) and such that $Q_i(b) = 0$ for $1 \leq i \leq n$. As $(f_1^r, \ldots, f_n^r) = A$, we may find $g_1, \ldots, g_n$ such that $\sum_{i=1}^n g_i f_i^r = 1$. It follows that $R = \sum_{i=1}^n g_i Q_i$ is a unitary polynomial of degree d annihilating b. This proves that b is algebraic over A.

To end the proof we still need to show that B is a finite type algebra. This follows from Lemma 3.16. □

Remark 3.44 — Let $f : X \to S$ be an entire (resp. finite) morphism. For every $S' \to S$, the base change $f' : X' = X \times_S S \to S'$ is also entire (resp. finite). Also the morphism $f_{red} : X_{red} \to S_{red}$ is entire (resp. finite).

Definition 3.45 — *A morphism $f : Y \to X$ is* dominant *if for every generic point η of X the fiber $f^{-1}(\eta)$ is non-empty.*

Theorem 3.46 — *Let $f : Y \to X$ be an entire and dominant morphism. Then f is surjective (i.e., the fiber $f^{-1}(x)$ is non-empty for any $x \in |X|$).*

Proof. Replacing f by f_{red}, we may assume that X and Y are reduced. Replacing X by an irreducible component and Y by its inverse image, we may assume that X is integral. Replacing Y by the closure of an element of $f^{-1}(\eta)$ (where η is the generic point of X), we may also assume that Y is integral.

Let $x \in X$. To see that $f^{-1}(x)$ is non-empty, we may replace X by an affine neighborhood of X. So we may assume that $X = \mathsf{Spec}(A)$ and $B = \mathsf{Spec}(B)$. Now, B is filtered union of its finitely generated A-subalgebra. Thus we may assume that B is also finitely generated.

Then B is a finitely generated A-module with no torsion. Denote $\mathfrak{p} = \mathfrak{m}_x$. Then $B_\mathfrak{p}$ is a non-zero finitely generated $A_\mathfrak{p}$-module. By Nakayama's Lemma, $\mathfrak{p}B_\mathfrak{p}$ is strictly contained in $B_\mathfrak{p}$. This implies that $B \otimes_A \kappa(x)$ is non-zero. □

Proposition 3.47 — *Let $f : Y \to X$ be a finite morphism. Then for every $x \in X$ the fiber $f^{-1}(x)$ is a discrete topological space. Moreover, for any $y \in f^{-1}(x)$, the extension $\kappa(x) \subset \kappa(y)$ is a finite extension.*

Proof. Let $Y_x = Y \times_X \mathsf{Spec}(\kappa(x))$. Then $Y_x \to \mathsf{Spec}(\kappa(x))$ is a finite morphism. This means that Y_x is the spectrum of a finite $\kappa(x)$-algebra E. Such an algebra is the direct sum f artinian algebra of finite type. This proves the claim. □

3.9. **Quasi-finite morphisms.**

Definition 3.48 — *A morphism of schemes $f : Y \to X$ is called* quasi-finite *if it is separated, of finite type and its fibers $f^{-1}(x)$ are finite sets for any $x \in X$.*

Lemma 3.49 — *A finite morphism of schemes is quasi-finite. A quasi-compact locally closed immersion is quasi-finite.*

Proof. A finite morphism is of finite type by definition. It is also separated as it is an affine morphism. Moreover, the fibers of a finite morphism are finite sets by Proposition 3.47.

For the second claim, we remark that a locally closed immersion is locally of finite type so that a quasi-compact locally closed immersion is of finite type. Moreover, the fibers of a locally closed immersion contain at most one element. □

Lemma 3.50 — *Let $f : Y \to X$ and $g : Z \to Y$ be two quasi-finite morphisms. Then $f \circ g$ is also quasi-finite.*

Proof. The composition $f \circ g$ is clearly separated and of finite type. Moreover, for $x \in X$, we have $(f \circ g)^{-1}(x) = \coprod_{y \in f^{-1}(x)} g^{-1}(y)$. This shows that the fibers of $f \circ g$ are finite. □

Proposition 3.51 — *Let $f : Y \to X$ be a quasi-finite morphism. Let x be a point of X. The $\kappa(x)$-scheme $Y_x = Y \times_X \mathsf{Spec}(\kappa(x))$ is finite, i.e., isomorphic to the spectrum of a finite $\kappa(x)$-algebra. In particular, the topological space $|Y_x|$ is finite and discrete. Moreover, for every $y \in Y$, $\kappa(f(y)) \subset \kappa(y)$ is a finite extension.*

Proof. We know that Y_x is a $\kappa(x)$-scheme of finite type having a topological space $|Y_x|$ consisting of finitely many elements. All the claims follow then from Lemma 3.52 below. □

Lemma 3.52 — *Let k be a field and X a k-scheme of finite type. Assume that $|X|$ has finitely many closed points. Then, X is isomorphic to the spectrum of a finite k-algebra. In particular, the topological space $|X|$ is finite and discrete.*

Proof. By Lemma 3.53 below, a point $x \in X$ is closed if and only if $\kappa(x)$ is a finite extension of k. In particular, if $U \subset X$ is an open subset, the k-scheme U has only finitely many closed points, namely those closed points of X which belong to U. We claim that we may assume that X admits exactly one closed point. Indeed, denote by $(x_\alpha)_{\alpha \in I}$ the finite family of closed points in X. As the set $\{x_\beta,\ \beta \neq \alpha\}$ is closed, we may find an affine open subset U_α such that $x_\alpha \in U_\alpha$ is the unique closed point of U_α. Assuming that the lemma is proved in this case, we get that $|U_\alpha| = \{x_\alpha\}$. As X can be covered by such U_α, we see that $|X| = \{x_\alpha,\ \alpha \in I\}$. Hence the topological space $|X|$ is discrete and finite. It follows that $X = \mathsf{Spec}(\prod_{\alpha \in I} \mathcal{O}_{X,x_\alpha})$. But, the k-algebra $\mathcal{O}_{X,x_\alpha}$ is of finite type and has exactly one prime ideal. Hence, it is artinian. This shows that $\mathcal{O}_{X,x_\alpha}$ is a finite dimensional k-vector space.

In the sequel, we assume that X is affine. As X is of finite type, we may assume that $X = \mathsf{Spec}(k[t_1, \ldots, t_n]/\mathfrak{a})$. We want to show that $A = k[t_1, \ldots, t_n]/\mathfrak{a}$ is finite dimensional k-vector space knowing that A posses finitely many maximal ideals. For this, we may replace k by its algebraic closure $\bar{k}$ and A by $\bar{A} = A \otimes_k \bar{k}$. In other words, we may assume that k is algebraically closed. By the remark above, we can also assume that A admits exactly one maximal ideal, i.e., A is local.

We denote by $\mathfrak{m}$ the maximal ideal in $k[t_1, \ldots, t_n]$ containing $\mathfrak{a}$. As k is algebraically closed we have $k = \kappa(\mathfrak{m})$. Denote by $u_i \in k$ the class of t_i in $A/\mathfrak{m}_\alpha$ modulo this isomorphism. It follows that $t_i - u_i \in \mathfrak{m}_\alpha$. We deduce immediately the equality of prime ideals

$$\mathfrak{m} = (t_1 - u_1, \ldots, t_n - u_n).$$

Making a change of variables, we may assume that $u_i = 0$.

Now, consider the k-algebra A_{t_i}. It is a finite type k-algebra with no maximal ideals. Hence, it is zero. This means that for N big enough, $t_i^N \in \mathfrak{a}$, i.e., there is a surjection

$$k[t_1]/t_1^N \otimes \cdots \otimes k[t_n]/t_n^N \twoheadrightarrow A.$$

This proves that A is a finite dimensional k-vector space. □

Lemma 3.53 — *Let k be a field and X a k-scheme locally of finite type. A point $x \in |X|$ is closed if and only if the residue field $\kappa(x)$ is a finite extension of k.*

Proof. We first prove the necessity of the condition. Assume that x is closed. By replacing X by an affine neighborhood of x, we may assume that $X = \mathsf{Spec}(A)$ is affine. The evaluation map $A \to \kappa(x)$ is surjective. This implies that $\kappa(x)$ is a finitely generated k-algebra which is a field. We must prove that every element of $\kappa(x)$ is algebraic over k. Assume to the contrary that there exists $a \in \kappa(x)$ which is transcendental over k, i.e., the morphism

$$k[t] \to \kappa(x)$$

sending t to a is injective. It follows that the field $k(t)$ is a subfield of $\kappa(x)$. Moreover, $\kappa(x)$ is a finitely generated $k(t)$-algebra. Repeating this procedure, we may construct a morphism of k-extensions

$$k(t_1, \ldots, t_n) \subset \kappa(x)$$

such that $\kappa(x)$ is an algebraic extension of $k(t_1, \ldots, t_n)$. Now let $u_1, \ldots, u_r$ be generators of the finite type k-algebra $\kappa(x)$. For $i \in [\![1, r]\!]$, choose polynomials $P_i \in k(t_1, \ldots, t_n)[s]$ such that $P_i(u_i) = 0$. We may find $f \in k[t_1, \ldots, t_n]$ such that all coefficients of P_i are in $k[t_1, \ldots, t_n]_f$. This implies that $\kappa(x)$ is a finite $k[t_1, \ldots, t_n]_f$-algebra which is dominant. Now, by Theorem 3.46, the morphism of schemes

$$\mathsf{Spec}(\kappa(x)) \to \mathsf{Spec}(k[t_1, \ldots, t_n]_f)$$

is surjective. As $|\mathsf{Spec}(\kappa(x))|$ contains only one element, we see that $k[t_1, \ldots, t_n]_f$ is a field. This is impossible unless $n = 0$.

Now, we assume that $\kappa(x)$ is a finite extension of k and we show that x is closed. Replacing X by $\overline{\{x\}}$ we may assume that x is the unique generic point of X. Assume that there exists $y \in X$ such that $y \neq x$ but $y \in \overline{\{x\}}$. We will get a contradiction. Replacing X by an affine neighborhood of y, we may assume that $X = \mathsf{Spec}(A)$ is affine. Then A is an integral k-algebra such that $\mathrm{Frac}(A)$ is a finite extension of k. Then A is a finite k-algebra which is integral. It is then already a field. This shows that X has only one point, which is a contradiction. □

Proposition 3.54 — *Let $f : Y \to X$ be a quasi-finite morphism and assume that X and Y have finitely many irreducible components. Then there exists a dense open subset $U \subset X$ such that $f^{-1}(U) \to U$ is a finite morphism.*

Proof. Using the easy fact that a morphism g of schemes is finite if and only if g_{red} is finite, we may assume that Y and X are reduced schemes. The problem is local with respect to X. Using the fact that X has finitely many irreducible component, we may assume that $X = \mathsf{Spec}(A)$ is the spectrum of an integral domain.

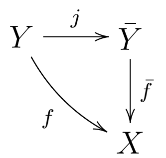
Denote by η the generic point of X and let $\{\xi_i;\ i \in I\} = f^{-1}(\eta)$. Each point ξ_i is a generic point of Y. Indeed, if we can write $\xi_i \in \overline{\{y\}}$ for some point $y \in Y$, then $\eta \in \overline{\{f(y)\}}$ which implies that $\eta = f(y)$. Hence $y \in f^{-1}(\eta)$. But by Proposition 3.51, we know that the topology on $f^{-1}(\eta)$ is the discrete one. This implies that $y = \xi_i$.

Now let $Y_0 \subset Y$ be an open subscheme containing the points ξ_i. Let $Z = Y \backslash Y_0$. Then $f(Z)$ does not contain the point η. By Lemma 3.57 (of the next paragraph), we may find an open neighborhood $U \subset X$ such that $f^{-1}(U) \cap Z = \emptyset$. This implies that we may replace Y by any dense open subscheme. We are thus reduced to the case where $Y = \mathsf{Spec}(B)$ is affine.

Now, to end the proof we argue as follows. By Proposition 3.51 the scheme-theoretic fiber $Y_\eta = \mathsf{Spec}(B \otimes_A \mathrm{Frac}(A))$ is a finite η-scheme. This implies that every element of B is algebraic over $\mathrm{Frac}(A)$. Fixing generators $b_1, \ldots, b_n$ of the A-algebra B, we get unitary polynomials $P_i \in \mathrm{Frac}(A)[s]$ such that $P_i(b_i) = 0$. We may find a non-zero $a \in A$ such that $P_i \in A_a[s]$. It follows then that b_i are algebraic over A_a. This implies that B_a is a finite A_a algebra. In other words, $f^{-1}(\mathrm{D}(a)) \to \mathrm{D}(a)$ is a finite morphism. □

The proof of the following theorem uses cohomological methods and will be given in the fifth chapter:

Zariski's Main Theorem — *Let $f : Y \to X$ be a quasi-finite morphism of schemes. Assume that X is noetherian. Then, there exists a commutative triangle*



with j an open immersion and $\bar{f}$ a finite morphism.

3.10. Constructible subsets and morphisms of finite type.

Definition 3.55 — *Let X be a noetherian scheme. A subset $T \subset |X|$ is called* constructible *if there exists a chain of subsets*

$$\emptyset = T_0 \subset T_1 \subset \ldots T_n = T$$

such that $T_i - T_{i-1}$ are locally closed. Equivalently, T is constructible if it is a disjoint union of locally closed subsets. We denote by $\mathrm{cst}(X)$ the set of constructible subsets of $|X|$.

Lemma 3.56 — $\mathrm{cst}(X)$ *is closed under finite union, finite intersection and complementation.*

Proof. We first show that $\mathrm{cst}(X)$ is closed under finite intersection. Let T and T' be two constructible subsets of X. We may write

$$T = \coprod_{i=1}^{m} S_i \qquad \text{and} \qquad T' = \coprod_{j=1}^{n} S'_j$$

with S_i and S'_j locally closed. Then

$$T \cap T' = \coprod_{(i,j)\in I\times J} S_i \cap S'_j.$$

So we only need to check that $S_i \cap S'_j$ is locally closed. But, we have

$$S_i \cap S'_j = (\overline{S_i} \cap \overline{S'_j})\backslash[(\overline{S_i}\backslash S_i) \cup (\overline{S'_j}\backslash S'_j)].$$

Next, we check that $\mathrm{cst}(X)$ is stable by complementation. Let T be a constructible subset and write $T = \coprod_{i\in I} S_i$ with S_i locally closed. Then $|X|\backslash T = \cap_{i\in I}|X|\backslash S_i$. As we now know that cst is stable by finite intersection, we are left to check that $|X|\backslash S_i$ is constructible. But we can write

$$|X|\backslash S_i = (|X|\backslash\overline{S_i}) \coprod (\overline{S_i}\backslash S_i).$$

Finally, stability of $\mathrm{cst}(X)$ by finite unions follows from the fact that $T \cup T' = |X|\backslash((X\backslash T) \cap (X\backslash T'))$. This finishes the proof of the lemma. □

Lemma 3.57 — *Let X be a scheme having finitely many irreducible components. Let $f : Y \to X$ be a quasi-compact morphism. Let $\eta \in X$ be a generic point. If $f^{-1}(\eta)$ is empty, then there exists an open neighborhood U of η in X such that $f^{-1}(U)$ is empty.*

Proof. We may replace X by an affine neighborhood of η and Y by the inverse image of this neighborhood. Doing so, we may assume that X is affine and irreducible. In this case, Y is quasi-compact. Writing Y as a union of finitely many affine opens, we see that it is sufficient to consider the case where Y is also affine. Finally, we may also replace Y and X by Y_{red} and X_{red}.

By the previous discussion, we only need to consider the case where $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(B)$ for some rings A and B with A integral and B reduced. The morphism f is induced by a morphism of rings $A \to B$. The condition that $f^{-1}(\eta) =$ is equivalent to the fact that the A-module $(A\backslash\{0\})^{-1}B$ is zero. This means that there exists a non-zero element $a \in A$ such that $a.1_B = 0$. It follows that $f^{-1}(\mathrm{D}(a)) = \emptyset$. This proves the lemma. □

Proposition 3.58 — *Let X be a scheme having finitely many irreducible components. Let $f : Y \to X$ be a morphism of finite type. Let $\eta \in X$ be a generic point such that $f^{-1}(\eta)$ is non-empty. Then the set $f(|Y|) \subset |X|$ contains an open neighborhood of η.*

Proof. Replacing X by an open neighborhood of η and Y by the inverse image of this neighborhood, we may assume that X is affine and irreducible. Replacing Y and X by Y_{red} and X_{red}, we may further more assume that X is integral.

The scheme $Y_\eta = Y \times_X \mathsf{Spec}(\kappa(\eta))$ is a finite type $\kappa(\eta)$-scheme which is not empty. Let ξ be a closed point of $Y_\eta \subset Y$. Then $f(\xi) = \eta$ and $\kappa(\eta) \subset \kappa(\xi)$ is a finite extension by Lemma 3.53. Replacing Y by the reduced closed subscheme $\overline{\{\xi\}}$, we may assume that Y is integral with generic point ξ such that $\kappa(\xi)$ is a finite extension of $\kappa(\eta)$.

We may replace Y by an open neighborhood of ξ. We are thus reduced to the case where $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(B)$ are integral and affine and $\mathrm{Frac}(A) \subset \mathrm{Frac}(B)$ is a finite extension. Let $b_1, \dots, b_n$ be generators of the A-algebra B. Then, there are monic polynomials $P_i \in \kappa(A)[s]$ such that $P_i(b_i) = 0$. Let $v \in A$ be such that the coefficients of P_i are in the subring $A_v \subset \mathrm{Frac}(A)$. Then, B_v is finite over A_v. By Theorem 3.46, $\mathsf{Spec}(B_v) \to \mathsf{Spec}(A_v)$ is surjective. This proves that $\mathrm{D}(v) \subset \mathsf{Spec}(A)$ is contained in the image of f. □

Theorem 3.59 — *Let X be a noetherian scheme and $f : Y \to X$ a morphism of finite type. Let T be a constructible subset of Y. Then $f(T)$ is a constructible subset of X.*

Proof. We immediately reduce to the case where $T = |Y|$. We will use noetherian induction on X, i.e., we assume that the theorem is true for strict closed subschemes of X. Let η be a generic point of X. If $f^{-1}(\eta)$ is empty, we may find an open subset $U \subset X$ such that $f^{-1}(U) = \emptyset$. This shows that f can be factored trough the closed subscheme $X \backslash U$. And the claim in this case follows by noetherian induction.

Let's now assume that $f^{-1}(\eta) \neq \emptyset$. We can find $U \subset X$ open such that $U \subset f(Y)$. It follows that $f(Y) = U \coprod f(Y \backslash f^{-1}(U))$. As $Y \backslash f^{-1}(U)$ factors trough $X \backslash U$, we know that $f(Y \backslash f^{-1}(U))$ is constructible by noetherian induction. The theorem is proven. □

3.11. Proper and universally closed morphisms.

Definition 3.60 — *Let $f : Y \to X$ be a morphism of schemes. We say that f is* closed *if for any closed subset $Z \subset Y$, $f(Z)$ is closed in X. We say that f is* universally closed *if for any X-scheme $X' \to X$, the base change $f' : Y \times_X X' \to X'$ is a closed morphism.*

Example 3.61 — A closed immersion $i : Y \to X$ is a universally closed morphism.

Lemma 3.62 — *A composition of two (universally) closed morphisms is a (universally) closed morphism.*

Lemma 3.63 — *Let $f : Y \to X$ be a morphism of finite type with X noetherian. To check that f is universally closed, it is sufficient to check that $f' : Y \times_X X' \to X'$ is closed for $X' \to X$ a finite type X-scheme.*

Proof. Assume that f is universally closed in the weak sense of the lemma. We want to show that f is universally closed in the sense of Definition 3.60. This is a local question over X. Thus, we may assume that $X = \mathsf{Spec}(A)$ is affine.

Let $X' \to X$ be any X-scheme and let's show that $f' : Y \times_X X' \to X'$ is closed. For this also, we may assume that $X' = \mathsf{Spec}(A')$ is affine. Set $Y' = Y \times_X X'$. Let $Z' \subset Y'$ be a closed subset and denote by $\mathcal{I}' \subset \mathcal{O}_{Y'}$ the sheaf of ideals of functions that vanishes at Z'. We denote by $\mathscr{F}$ the set of finitely generated sub-ideals of $\mathcal{I}'$. By Lemma 3.64 below, we know that $\mathcal{I}' = \cup_{\mathcal{J} \in \mathscr{F}} \mathcal{J}$. It follows that $Z' = \cap_{\mathcal{J} \in \mathscr{F}} \mathcal{Z}(\mathcal{J})$ (where, recall, $\mathcal{Z}(\mathcal{J})$ is the closed subset of Y' whose points are $y' \in Y'$ such that $\mathcal{J}_{y'} \subset \mathfrak{m}_{y'} \subset \mathcal{O}_{Y',y'}$).

We claim that $f(Z') = \cap_{\mathcal{J} \in \mathscr{F}} f(\mathcal{Z}(\mathcal{J}))$. The inclusion $\subset$ is clear. On the other hand, assume that $x' \in X'$ is in the image of all $\mathcal{Z}(\mathcal{J})$ for $\mathcal{J} \in \mathscr{F}$. This means that the closed subset $\mathcal{Z}(\mathcal{J})_{x'} \subset Y'_{x'}$ is not empty. As $Y'_{x'}$ is a noetherian scheme and the ordered set $\mathscr{F}$ is filtered, we deduce that there exists $\mathcal{J}_0 \in \mathscr{F}$ such that for every $\mathcal{J} \in \mathscr{F}$ contaning $\mathcal{J}_0$, we have $\mathcal{Z}(\mathcal{J})_{x'} = \mathcal{Z}(\mathcal{J}_0)_{x'}$. This implies that $Z'_{x'} = \mathcal{Z}(\mathcal{J}_0)_{x'}$. This shows that $Z'_{x'}$ is not empty and thus $x' \in f(Z')$.

So, in order to prove the lemma, we may assume that Z' is the zero set of a finitely generated ideal of $\mathcal{O}_{Y'}$. It is easy to see that there exists a finitely generated sub-A-algebra $B \subset A'$ and an ideal $\mathcal{I} \subset \mathcal{O}_{Y \otimes_A B}$ such that Z' is the inverse image of $\mathcal{Z}(\mathcal{I})$ by the obvious morphism $Y' = Y \otimes_A A' \to Y \otimes_A B$. Replacing X by $\mathsf{Spec}(B)$ and Y by $Y' \otimes_A B$, we may assume that Z' is the inverse image of a closed subset $Z \subset Y$ by the obvious morphism $Y' \to Y$.

Now, we are in the following situation. We have a cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{f} & X \end{array}$$

a subscheme $Z \subset Y$ and we want to prove that $f'(b^{-1}(Z))$ is closed. For this it is sufficient to show that $f'(b^{-1}(Z)) = a^{-1}(f(Z))$ (remember that f is a closed morphism). We obviously have $f'(b^{-1}(Z)) \subset a^{-1}(f(Z))$. Let $x' \in a^{-1}(f(Z))$. We can find $z \in Z$ such that $f(z) = a(x')$. By Proposition 2.58, there exists $y' \in Y'$ such that $b(y') = z$ and $f'(y') = x'$. This proves that $x' \in f'(b^{-1}(Z))$. □

Lemma 3.64 — *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Then every finite family of sections of $\mathcal{M}$ is contained in a finitely generated quasi-coherent $\mathcal{O}_X$-submodule.*

Proof. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are finitely generated sub-modules of $\mathcal{M}$, then $\mathcal{M}_1 + \mathcal{M}_2 \subset \mathcal{M}$ is also a finitely generated submodule of $\mathcal{M}$. Hence, it is sufficient to construct a finitely generated sub-module $\mathcal{N} \subset \mathcal{M}$ such that $a \in \mathcal{N}(U)$ for a given section $a \in \mathcal{M}(U)$ over an open sub-scheme $U \subset X$. We break the proof in two steps.

Step 1: Here we consider the case where $X = \mathsf{Spec}(A)$ is affine and $U \subset \mathsf{Spec}(A)$ an arbitrary quasi-compact open subset. Denote by M the A-module of global sections $\mathcal{M}(X)$. As $\mathcal{M} \simeq \widetilde{M}$, we may replace $\mathcal{M}$ by $\widetilde{M}$.

Let $(f_i)_{i \in I}$ be a finite family of elements in A such that $U = \cup_{i \in I} \mathrm{D}(f_i)$. We denote $a_i = a_{|\mathrm{D}(f_i)}$. We can write $a_i = \dfrac{m_i}{f_i^{n_i}}$ with $m_i \in M$ and $n_i \in \mathbb{N}$. Now let $N = \sum_{i \in I} A.m_i \subset M$. By construction, we have $a_i \in \widetilde{N}(\mathrm{D}(f_i))$. As $\widetilde{N}$ is a subsheaf of $\widetilde{M}$, the sections a_i glue in $\widetilde{N}$. In other words, we have $a \in \widetilde{N}(U)$.

Step 2: Now we turn to the general case. We fix a finite open covering $(X_i)_{i \in I}$ of X by affine subschemes. We use induction on the number of elements in I. When I contains 1 element, the result is known by the step before. Let $i_0 \in I$ and set $J = I \backslash \{i_0\}$. Let $X' = \cup_{j \in J} X_j$, $\mathcal{M}' = \mathcal{M}_{|X'}$, $U' = U \cap X'$ and $a' = a_{|U'}$. By the induction hypothesis, there exists $\mathcal{N}' \subset \mathcal{M}'$ finitely generated and containing a'.

To prove the lemma, it suffices to construct a finitely generated quasi-coherent sub-module $\mathcal{P} \subset \mathcal{M}_{|X_{i_0}}$ satisfying the following two conditions:

- $\mathcal{P}_{|X_{i_0} \cap X'} = \mathcal{N}'_{|X_{i_0} \cap X'}$,
- $a_{|U \cap X_{i_0}} \in \mathcal{P}(U \cap X_{i_0})$.

Let E be the fiber product of the diagram of $A_{i_0} = \mathcal{O}_X(X_{i_0})$-modules

$$\begin{array}{ccc} & & \mathcal{M}(X_{i_0}) \\ & & \downarrow \\ \mathcal{N}(X' \cap X_{i_0}) & \hookrightarrow & \mathcal{M}(X' \cap X_{i_0}) \end{array}$$

This is a sub-module of $\mathcal{M}(X_{i_0})$. $\widetilde{E}_{|X' \cap X_{i_0}} = \mathcal{N}_{|X' \cap X_{i_0}}$ and $\widetilde{E}(U \cap X_{i_0})$ contains $a_{|U \cap X_{i_0}}$. We may replace E by a finitely generated A_{i_0}-submodule P and still have the same properties. We then take $\mathcal{P} = \widetilde{P}$. □

Definition 3.65 — *Let $f : Y \to X$ be a morphism of schemes. We say that f is* proper *if it is of finite type, separated and universally closed.*

Proposition 3.66 — *A finite morphism $f : Y \to X$ is proper.*

Proof. It suffices to show that f is closed. The problem is local on X. So we may assume that $X = \mathsf{Spec}(A)$ is affine. It follows that $Y = \mathsf{Spec}(B)$ with B a finite A-algebra. Let $b_1, \ldots, b_n$ be generators of the A-algebra B and $P_1, \ldots, P_n \in A[s]$ be unitary polynomials which are zero on the b_i. Then we get a surjective morphism

$$C = A[s_1]/P_1 \otimes_A \cdots \otimes_A A[s_n]/P_n \twoheadrightarrow B.$$

It follows that $\mathsf{Spec}(B)$ is a closed subscheme of the finitely presented $\mathsf{Spec}(A)$-scheme $\mathsf{Spec}(C)$. Thus, we may assume that B is finitely presented. Thus, we can find a subring $A_0 \subset A$ of finite type over $\mathbb{Z}$ and a finite A_0-algebra B_0 such that $B = A \otimes_{A_0} B_0$. By Lemma 3.63 we are reduced to the case where A is noetherian and it suffices to show that $f(Y)$ is closed. As Y has finitely many irreducible component, we can assume that Y is integral. By Theorem 3.46, $f(Y)$ is the closure of the image by f of the generic point of Y. This ends the proof. □

Proposition 3.67 — *Consider a commutative triangle*

$$\begin{array}{ccc} Y & \xrightarrow{t} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & X \end{array}$$

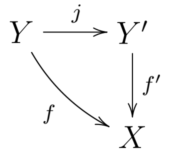
with f a proper (resp. a universally closed) morphism and g a separated morphism. Then t is proper (resp. universally closed).

Proof. We have a commutative diagram with a cartesian square

$$\begin{array}{ccccc} Y & \xrightarrow{(\mathrm{id},t)} & Y \times_X Z & \xrightarrow{pr_2} & Z \\ \downarrow & & \downarrow & & \\ Z & \xrightarrow{\Delta} & Z \times_X Z & & \end{array} \qquad (\text{with } t = pr_2 \circ (\mathrm{id},t) : Y \to Z)$$

As g is separated, we deduce that (id, t) is a closed immersion. Moreover, $pr_2 : Y \times_X Z \to Z$ is proper (resp. universally closed). This proves the proposition. □

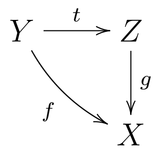
Corollary 3.68 — *Consider a commutative triangle*



with j a dense open immersion, f a universally closed morphism and f' a separated morphism. Then j is an isomorphism.

Proof. Indeed, j is universally closed. In particular, its image is closed. As j is dense, we see that $j(Y) = Y'$. This proves our claim. □

Lemma 3.69 — *Consider a commutative triangle*



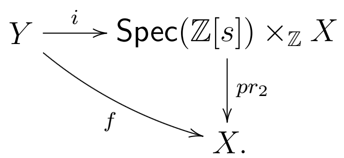
with f proper (resp. universally closed) and g separated. If t is dominant, then g is proper (resp. universally closed).

Proof. First, note that t is surjective. Indeed, t being dominant, we know that $t(Y)$ is dense. But as t is closed, $t(Y)$ is a closed subset of Z.

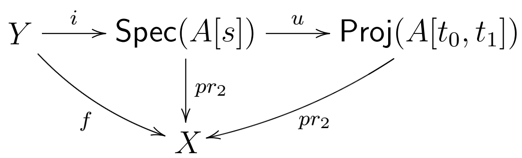
As being surjective, is stable by base change, we only need to check that g is closed. Let $A \subset Z$ be a closed subset. As t is surjective, we have $A = tt^{-1}(A)$. Our claim follows then from $g(A) = gtt^{-1}(A) = f(t^{-1}(A))$, that $t^{-1}(A)$ is closed and that f is proper. □

Proposition 3.70 — *Let $f : Y \to X$ be an affine morphism which is universally closed. Then f is entire. (In particular, if f is of finite type, then f is finite.)*

Proof. We may assume that $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(B)$ for a ring A and an A-algebra B. We must show that every element of B is algebraic over A. Let $b \in B$. Let $B_0 \subset B$ be the subring of B generated by b. As $\mathsf{Spec}(B) \to \mathsf{Spec}(B_0)$ is dominant, we know by Lemma 3.69 that $\mathsf{Spec}(B_0) \to \mathsf{Spec}(A)$ is universally closed. Hence, we may assume that B is generated by b. In other words, we have a commutative triangle



Consider now the commutative diagram



with $s = t_1/t_0$. Applying Proposition 3.67, we see that $u \circ i$ must be a closed immersion. Therefore, there exists a homogenous ideal $I \subset A[t_0, t_1]$ such that

$Z_+(I) = ui(Y)$. As $ui(Y) \cap \mathcal{Z}_+(t_0)$, we see that $(t_0, t_1) \subset \sqrt{I + (t_0)}$. This means that there exists $n \in \mathbb{N}$ such that $t_1^n \in I + (t_0)$, i.e., there exists $P \in I$ of the form $t_1^n + P(t_0, t_1)t_0$ (with P homogenous of degree $n-1$). This means that our $b \in B$ satisfies to the equation $b^n + P(1, b) = 0$. This proves that b is algebraic. □

Corollary 3.71 — *Let $f : Y \to X = \mathsf{Spec}(A)$ be a universally closed morphism. Then the A-algebra $\mathcal{O}_Y(|Y|)$ is entire.*

Proof. Indeed, $Y \to \mathsf{Spec}(\mathcal{O}_Y(|Y|))$ is dominant. It follows that $\mathsf{Spec}(\mathcal{O}_Y(|Y|)) \to \mathsf{Spec}(A)$ is universally closed. □

3.12. Discrete valuation rings.

Recall that a *discrete valuation ring* (d.v.r.) is a ring A satisfying to one of the equivalent conditions:

(1) A is a principal ideal domain which is local,
(2) A is integral and there is a discrete valuation $\nu : \mathrm{Frac}(A) \to \mathbb{N} \cup \{\infty\}$ such that $A = \{b \in \mathrm{Frac}(A);\ \nu(a) \geq 0\}$.

(Recall that a discrete valuation on a field K is an application $\nu : K \to \mathbb{N} \cup \{\infty\}$ such that $\nu(a) = \infty$ if and only if $a = 0$, $\nu(ab) = \nu(a) + \nu(b)$ and $\nu(a + b) \geq \min(\nu(a), \nu(b))$.)

It is clear that condition (2) implies (1). Assume that (1) is true and A is not a field. Then A is noetherian and has a unique maximal ideal $\mathfrak{m}_A$ generated by one element π_A (called a uniformizing element). A non-zero element $a \in \mathrm{Frac}(A)$ can be written in a unique way as $a = u\pi_A^n$ where $u \in A^\times$ and $n \in \mathbb{Z}$. Moreover, we can put $\nu(a) = n\nu(\pi_A)$ for any fixed value of $\nu(\pi_A)$.

Definition 3.72 — *Let A be an integral domain. A prime ideal $\mathfrak{p} \subset A$ is called of codimension ≤ 1 if the condition $\mathfrak{q} \subset \mathfrak{p}$ for $\mathfrak{q}$ implies that $\mathfrak{q} \in \{(0), \mathfrak{p}\}$. We say that $\mathfrak{p}$ is of codimension 1 if moreover it is different from (0).*

Proposition 3.73 — *The following conditions for a ring A are equivalent*

(1) *A is a discrete valuation ring which is not a field.*
(2) *A is noetherian, local, normal and its maximal ideal is of codimension 1.*

Proof. The implication (1) ⇒ (2). We concentrate on the other direction. Let A be a normal, local ring whose maximal ideal $\mathfrak{m}$ is of codimension 1. Let K be the fraction field of A. An A-submodule of K which is finitely generated and non-zero is called a fractional ideal. Let $U = \{a \in K;\ a.\mathfrak{m} \subset \mathfrak{m}\}$. It is clear that U is a sub-A-algebra of K. Moreover, U is a fractional ideal. Indeed, if $x \in \mathfrak{m}$ is a non-zero element, we have $U \subset \{a \in K, a(x) \subset A\} = A.x^{-1}$. The claim follows as A is noetherian.

As U is a finitely generated A-module, we get that U is a finite A-algebra. As A is assumed to be normal, we get finally that

$$\{a \in K;\ a.\mathfrak{m} \subset \mathfrak{m}\} = A.$$

Now in general, for M and N two fractional ideals in K, we set $M : N = \{a \in K;\ aN \subset M\}$. This is again a fractional ideal. Let $\mathcal{S}$ be the set of ideals $I \subset A$ such that $A : I$ contains strictly A. If $b \in \mathfrak{m}$ is non-zero, we clearly have $(b) \in \mathcal{S}$. Thus $\mathcal{S}$ is non-empty. As A is noetherian, every chain in $\mathcal{S}$ is stationary. Hence, there exists a maximal element $\mathfrak{n} \in \mathcal{S}$. Let's show that $\mathfrak{n}$ is a prime ideal. Let a and

b be two elements in A such that $ab \in \mathfrak{n}$. Fix $c \in A : \mathfrak{n}$ which in not in A. Assume that $a \notin \mathfrak{n}$. As $\mathfrak{n}$ is maximal, we know that $(a) + \mathfrak{n} \notin \mathcal{S}$. This implies that $ca \notin A$, because otherwise c would be in $A : ((a) + \mathfrak{n})$. Now, $cab \in \mathfrak{n}$. This implies that $ca \in A : ((b) + \mathfrak{n})$. The maximality of $\mathfrak{n}$ forces that $b \in \mathfrak{n}$.

To end the proof, we use the fact that $\mathfrak{m}$ is the only non-zero prime ideal. This implies that $\mathfrak{m} = \mathfrak{n}$. Hence, $\mathfrak{m}' = A : \mathfrak{m}$ contains A strictly. The fractional ideal $\mathfrak{a} = \mathfrak{m}\mathfrak{m}'$ is contained in A. It cannot be strict because otherwise, $\mathfrak{a} \subset \mathfrak{m}$ and this gives that $\mathfrak{m}' = \mathfrak{m} : \mathfrak{m} = A$ which is a contradiction. Therefore, we have $\mathfrak{m}'\mathfrak{m} = A$. We can find $a_1, \ldots, a_n \in \mathfrak{m}$ and $b_1, \ldots, b_n \in \mathfrak{m}'$ such that $1 = \sum_{i=1}^n a_i b_i$. As A is local, there is i_0 such that $a_{i_0} b_{i_0}$ is invertible. This proves that there is $a \in \mathfrak{m}$ and $b \in \mathfrak{m}'$ such that $ab = 1$. This means that multiplication by b induces an isomorphism between $\mathfrak{m}$ and A. Hence $\mathfrak{m}$ is principal.

It is now easy to finish the proof of the proposition. Let π be a generator of the maximal ideal $\mathfrak{m} \subset A$. Let $a \in A$. We can write $a = u\pi^n$ with u invertible. This easily implies that any ideal of A is principal. □

Corollary 3.74 — *Let A be noetherian normal integral domain. Let $\mathfrak{p} \subset A$ be a prime ideal of codimension 1. Then $A_{\mathfrak{p}}$ is a discrete valuation ring.*

Theorem 3.75 — *Let A be an integral noetherian ring such that every maximal ideal of A is of codimension ≤ 1. Let $L/K = \mathrm{Frac}(A)$ be a finite extension. Then the normalization of A in L is a noetherian ring.*

We first prove a lemma:

Lemma 3.76 — *Let M be an A-module such that $M \otimes_A K$ has finite dimension d. For $a \in A \backslash \{0\}$, we have $\mathrm{lg}(M/aM) \leq d.\mathrm{lg}(A/aA)$.*

Proof. We first consider the case when M is finitely generated. We can find a free sub-module $N \subset M$ of rank d. Then M/N is a torsion finitely generated A-module. Hence it has finite length. For $n \in \mathbb{N}$, we have an exact sequence

$$N/a^n N \to M/a^n M \to (M/N) \otimes_A A/a \to 0.$$

We get in this way the inequalities

$$\mathrm{lg}(M/a^n M) \leq nd.\mathrm{lg}(A/aA) + \mathrm{lg}(M/N).$$

Moreover, multiplications by a^i induces isomorphisms $M/aM \simeq a^i M/a^{i+1}M$. This shows that $\mathrm{lg}(M/a^n M) = n.\mathrm{lg}(M/aM)$. We finally get

$$\mathrm{lg}(M/aM) \leq d.\mathrm{lg}(A/aA) + \frac{1}{n}\mathrm{lg}(M/N).$$

Letting n tends to infinity yields the desired result.

For the general case, we argue as follows. Let $\mathcal{S}$ be the set of finitely generated sub-A-modules $N \subset M$ such that $N \otimes_A K = M \otimes_A K$. The A/aA-module M/aM is the filtered union of the images of N/aN, i.e.,

$$M/aM = \bigcup_{N \in \mathcal{S}} (N + aM/aM)$$

As $\mathrm{lg}(N + aM/aM) = \mathrm{lg}(N/N \cap aM) \leq \mathrm{lg}(N/aN) \leq d.\mathrm{lg}(A/aA)$ we get the desired statement. □

Proof of Theorem 3.75. Let B be the integral closure of L with respect to A. Then B is an A-module such that $B \otimes_A K = L$ is of finite dimension. It follows that for every $a \in A$, the A-module (and thus the B-module) B/aB is of finite length.

Now let $\mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \cdots \subset \mathfrak{b}_n \subset \dots$ be an infinite chain of non-zero ideals in B. We know that $\mathfrak{b}_1 \cap A$ is non-zero. Let a be a non-zero element of this intersection. As B/aB has finite length, we see that the sequence of A-modules $\mathfrak{b}_i B/aB \subset \mathfrak{b}_{i+1}B/aB$ is stationary (say for $i \geq N$). It follows that $\mathfrak{b}_{N+i} = \mathfrak{b}_N$ for all $i \in \mathbb{N}$. This ends the proof. □

Proposition 3.77 — *Let A be a noetherian integral domain and $\mathfrak{p}$ a non-zero prime ideal of A. There exists a d.v.r. R and a dominant morphism $A \to R$ such that $\mathfrak{p}$ is the inverse image of the maximal ideal of R.*

Proof. There exists $a \in \mathfrak{p}$ such that $a^n \notin \mathfrak{p}^{n+1}$ for all n. Indeed, if $a_1, \dots, a_r$ generates $\mathfrak{p}$ and if $a_i^{n_i} \in \mathfrak{p}^{n_i+1}$ we get with $N = \sum_i n_i$ $\mathfrak{p}^N = \mathfrak{p}^{N+1}$. As A is noetherrian, this implies that $\mathfrak{p}^N = 0$. But A is integral and $\mathfrak{p}$ is non-zero.

Fix $a \in \mathfrak{p}$ as before and consider $B \subset \mathrm{Frac}(A)$ the subring $\bigcup_{n \in \mathbb{N}} a^{n-1}\mathfrak{p}^n$. Then $a \in B$ is a not invertible. Indeed, if we can write $a^{-1} = m.a^{-n}$ with $m \in \mathfrak{p}^n$, then we would have $a^{n+1} \in \mathfrak{p}^n$ which contradicts our choice of a.

Moreover, by construction $\mathfrak{p}B = Ba$. Let $\mathfrak{q} \subset B$ be a minimal prime containing a. By the lemma below, $\mathfrak{q}$ is of codimension 1. Let C be the normalization of $B_\mathfrak{q}$ and $\mathfrak{n} \subset C$ a maximal ideal. Then $R = C_\mathfrak{n}$ satisfies the needed properties. □

Lemma 3.78 — *Let A be an integral noetherian domain, $a \in A$ a non-zero element and $a \in \mathfrak{p} \subset A$ a minimal prime ideal containing a. Then $\mathfrak{p}$ is of codimension 1.*

Proof. By contradiction, we assume that there is a strict non-zero prime ideal $\mathfrak{q} \subset \mathfrak{p}$. Then $a \notin \mathfrak{q}$. Replacing A by $A_\mathfrak{p}$ we reduce to the case where A is local and $\mathfrak{p}$ is its maximal ideal.

Let $y \in \mathfrak{q}$ be a non-zero element. We will show that the sequence of integers $\lg(A/(x^m, y))$ stabilize (remark that $A/(x^m)$ is an artinian ring). Indeed, we have

$$\lg(A/(x^m,y)) = \lg(A/(x^m)) - \lg((x^m,y)/(x^m))$$

$$= \lg(A/(x^m)) - \lg((y)/(y)\cap(x^m))$$

$$= \lg(A/(x^m)) - \lg\left(\frac{A}{y^{-1}[(y)\cap(x^m)]}\right)$$

$$= \lg\left(\frac{y^{-1}[(y)\cap(x^m)]}{(x^m)}\right) = \lg\left(\frac{(y)\cap(x^m)}{(yx^m)}\right) = \lg\left(\frac{x^{-m}[(y)\cap(x^m)]}{(y)}\right)$$

As $x^{-m}[(y)\cap(x^m)]$ is an increasing family of ideals we get our claim from the fact that A is noetherian.

It follows that for some $m \in \mathbb{N}$ we have $(x^m, y) = (x^{m+1}, y)$. This means that we can write $x^m = ay + bx^{m+1}$. We get in this way that $x^m(1+bx) \in \mathfrak{q}$. As A is local with maximal ideal $\mathfrak{p}$ we see that $1 + bx$ is invertible. This gives that $x^m \in \mathfrak{q}$ which is a contradiction. □

3.13. **Valuative criteria.**

Let A be d.v.r. which is not a field. The scheme $\mathsf{Spec}(A)$ has exactly two points, namely one closed point and one generic point. When no confusion will arise these two points are denoted by o and η respectively. From Proposition 3.77 we get immediately the following fact:

COROLLARY 3.79 — *Let X be a noetherian scheme, $x \in X$ and $y \in \overline{\{x\}}$. be two points. Then there exists a d.v.r. A and a morphism of schemes $\mathsf{Spec}(A) \to X$ sending η to x and o to y.*

We have the following result:

PROPOSITION 3.80 — *Let X be a noetherian scheme and $T \subset |X|$ a constructible subset. The following conditions are equivalent:*

(1) *T is closed.*
(2) *For every d.v.r. A the following holds. For every $a : \mathsf{Spec}(A) \to X$ such that $a(\eta) \in T$ we also have $a(o) \in T$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious as we have $a(o) \in \overline{a(\eta)}$. To show the converse, we write $T = \coprod_{i\in I} S_i$ with I finite and S_i a locally closed subset. It suffices to show that $\overline{S_i} \subset T$. Let $y \in \overline{S_i}$. There exists a generic point $x \in \overline{S}_i$ such that $y \in \overline{\{x\}}$. As S_i is open and dense in $\overline{S_i}$ we have $x \in S_i$. Now by Corollary 3.79 we can find $a : \mathsf{Spec}(A) \to X$ with A a d.v.r. such that $a(\eta) = x$. and $a(o) = y$. By (2) we then get that $y \in T$. This finishes the proof of the proposition. □

DEFINITION 3.81 — *Let $f : Y \to X$ be a morphism of finite type with X noetherian. We say that f satisfies to the* valuative criterion for separatedness *if the following holds for every d.v.r. A. For every double-arrow*

$$\mathsf{Spec}(A) \underset{a_2}{\overset{a_1}{\rightrightarrows}} Y$$

such that $a_1 \circ (\eta \subset \mathsf{Spec}(A)) = a_2 \circ (\eta \subset \mathsf{Spec}(A))$ and $f \circ a_1 = \circ a_2$, we have $a_1 = a_2$.

THEOREM 3.82 — *Let $f : Y \to X$ be a finite type morphism with X a noetherian scheme. Then f is separated if and only if it satisfies to the valuative criterion for separatedness.*

Proof. By definition, f is separated if and only if the diagonal $\Delta_f(Y) \subset Y \times_X Y$ is a closed subset. By Proposition 3.80 the is equivalent to the following. For every d.v.r. A and every morphism $a : \mathsf{Spec}(A) \to Y \times_X Y$ such that $a(\eta) \in \Delta_f(Y)$ we have $a(\mathsf{Spec}(A)) \subset \Delta_f(Y)$. Let $a_i = pr_i \circ a$ where $a_i : Y \times_X Y \to Y$ is the projection to the i-th factor. Then the condition $a(\eta) \in \Delta_f(Y)$ is equivalent to $(a_1)_{|\eta} = (a_2)_{|\eta}$. Similarly the condition $a(\mathsf{Spec}(A)) \subset \Delta_f(Y)$ is equivalent to $a_1 = a_2$. This proves the theorem. □

DEFINITION 3.83 — *Let $f : Y \to X$ be a morphism of finite type with X noetherian. We say that f satisfies to the valuative criterion for properness if the* *following holds for any d.v.r. A. For every commutative solid square*

$$\begin{array}{ccc} \eta & \longrightarrow & Y \\ \downarrow & \overset{r}{\nearrow} & \downarrow f \\ \mathsf{Spec}(A) & \longrightarrow & X \end{array}$$

there is a unique morphism r (called a lift*) making commutative the hole diagram.*

Theorem 3.84 — *Let $f : Y \to X$ be a finite type morphism with X noetherian. Then f is proper if and only if it satisfies to the valuative criterion for properness.*

Proof. The condition is necessary. Indeed assume that f is proper and consider a commutative square as in Definition 3.83. The we can form the commutative triangle

$$\begin{array}{ccc} \eta & \xrightarrow{u} & Y \times_X \mathsf{Spec}(A) \\ & \searrow & \downarrow f' \\ & & \mathsf{Spec}(A) \end{array}$$

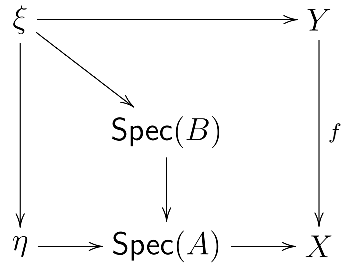
Let $Z \subset Y \times_X \mathsf{Spec}(A)$ be the closure of $u(\eta)$ endowed with the structure of a reduced subscheme. As f' is a closed morphism, we see that $f(Z') = \mathsf{Spec}(A)$ as $f(Z')$ contains η. This shows that $Z \to \mathsf{Spec}(A)$ is surjective. Let $z \in Z$ be a point above $o \in \mathsf{Spec}(A)$. Then $\mathcal{O}_{Z,z}$ is a subring of $\mathrm{Frac}(A)$ which is not a field and which contains A. As A is a d.v.r. we get $\mathcal{O}_{Z,z} = A$. The composition

$$r : \mathsf{Spec}(A) \simeq Z_z \to Y \times_X \mathsf{Spec}(A) \to Y$$

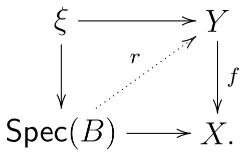
gives the lifting we are looking for. The uniqueness of such a lifting follows from the fact that f is separated.

Now we show that f is proper if it satisfies the valuative criterion for properness. The fact that f is separated follows from the uniqueness of the lifting in Definition 3.83. Remark also that if f satisfies the valuative criterion for properness then so does every base change of f by a finite type X-scheme. Thus, we are reduced (by Lemma 3.63) to show that f is a closed morphism. By replacing Y by its closed subschemes, we are finally reduced to show that $f(Y)$ is a closed subset.

To do this, we use Proposition 3.80 and check the condition (2) for $f(Y)$. Let A be a d.v.r and $a : \mathsf{Spec}(A) \to X$ be a morphism such that $a(\eta) \in f(Y)$. This means that the $\kappa(\eta)$-scheme $Y \times_X \eta$ is non-empty. Let $\xi \in Y \times_X \eta$ be a closed point. As the $Y \times_X \eta$ is of finite type over η we know that $\kappa(\xi)/\kappa(\eta)$ is a finite extension. Let $B' \subset \kappa(\xi)$ be the sub-A-algebra of A-algebraic elements. Let $\mathfrak{n}' \subset B'$ be a maximal ideal and set $B = B'_{\mathfrak{n}'}$. Then B is a d.v.r. containing A and such that $\mathrm{Frac}(B)/\mathrm{Frac}(A)$ is a finite extension. Moreover, we have a point $\xi \in Y \times_X \mathsf{Spec}(A)(\mathrm{Frac}(B))$ whose image is $u(\eta)$. We now form the composition $\xi \to Y \times_X \mathsf{Spec}(A) \to Y$. We thus have a commutative diagram



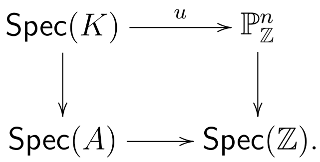
and in particular a commutative solid square



By the valuative criterion for properness we may find a lifting r. This shows that $f(Y)$ contains the image of Spec(B) which is also the image of Spec(A). This finishes the proof of the theorem. □

Corollary 3.85 — *For every n the $\mathbb{Z}$-scheme $p : \mathbb{P}^n_{\mathbb{Z}} \to \mathsf{Spec}(\mathbb{Z})$ is proper.*

Proof. We check the valuative criterion for properness for p. Let A be a d.v.r with fraction field K and consider a commutative diagram



The point $u \in \mathbb{P}^n_{\mathbb{Z}}(K)$ corresponds to an element of $(K^{n+1}\backslash\{0\})/K^\times$ and it can be written as a symbol $[a_0 : \cdots : a_n]$ where at least one of the a_i is non-zero. Let π be a uniformizing element of A. Let $m = \min(\nu(a_i))$ where ν is the discrete valuation of K such that $\nu(\pi) = 1$ and $\mathcal{O}_\nu = A$. Multiplying by π^{-m}, we may assume that $a_i \in A$ with at least one of them in $A^\times$. In this case the symbol $[a_0 : \cdots : a_n]$ gives a point in $\mathbb{P}^n(A)$ which induces the lifting we are looking for. □

3.14. **Open and flat morphisms.**

Definition 3.86 — *Let $f : Y \to X$ be a morphism of schemes. We say that f is* open *if for any open subscheme $V \subset Y$, the subset $f(V) \subset X$ is open. We say that f is* universally open *if for any X-scheme X', the base change $f' : Y \times_X X' \to X'$ is an open morphism.*

Lemma 3.87 — *Composition of open (resp. universally open) morphisms is an open (resp. universally open) morphism. An open immersion is a universally open morphism.*

Lemma 3.88 — *Let $f : Y \to X$ be a morphism of schemes and $(Y_j)_{j\in J}$ an open covering of Y. Let $f_j = f_{|Y_j}$. Then f is open (resp. universally open) if and only if f_j are open (resp. universally open) for every $j \in J$.*

Lemma 3.89 — *Let $f : Y \to X$ be a finite type morphism with X a noetherian scheme. To check that f is universally open it suffices to show that for every finite type X-scheme X', the base change $f' : Y \times_X X' \to X'$ is an open morphism.*

Proof. Let X' be a general X-scheme and let's show that $f' : Y' = Y \times_X X' \to X'$ is open (assuming that we know this if X' was of finite type over X). We reduce to the case where $Y = \mathsf{Spec}(B)$, $X = \mathsf{Spec}(A)$ and $X' = \mathsf{Spec}(A')$ are affine schemes. If suffices to show that $f'(\mathrm{D}(b'))$ is open for $b' \in B' = B \otimes_A A'$. But there is a finite type sub-A-algebra $A'' \subset A'$ and $b'' \in B'' = B \otimes_A A''$ which is mapped to b' by $B'' \to B'$. We thus have a cartesian square

$$\begin{array}{ccc} \mathsf{Spec}(B') & \xrightarrow{f'} & \mathsf{Spec}(A') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{Spec}(B'') & \xrightarrow{f''} & \mathsf{Spec}(B') \end{array}$$

and $\mathrm{D}(b') = p^{-1}(\mathrm{D}(b''))$. Thus we get $f'(\mathrm{D}(b')) = f'p^{-1}(\mathrm{D}(b'')) = q^{-1}f''(\mathrm{D}(b''))$. But we know that $f''(\mathrm{D}(b''))$ is open. This ends the proof of the lemma. □

In this paragraph we will give a large family of open morphisms. For this we need the notion of flatness from commutative algebra.

Definition 3.90 — *Let A be a ring. An A-module F is called* flat *if for any short exact sequence of A-modules*

$$0 \to M' \to M \to M'' \to 0$$

the sequence

$$0 \to M' \otimes_A F \to M \otimes_A F \to M'' \otimes_A F \to 0$$

is also a short exact sequence.

An A-algebra B is called a flat *A-algebra if B is flat as an A-module.*

The following property is obvious.

Lemma 3.91 — *The A-module A is flat. Flat A-modules are stable by direct sums (possibly infinite) and by tensor product.*

Lemma 3.92 — *Let A be a ring and F an A-module. Let B an A-algebra. Then $G = F \otimes_A B$ is a B-module in a natural way. Moreover, if F is a flat A-module then G is a flat B-module.*

Proof. Indeed, let

$$0 \to N' \to N \to N'' \to 0$$

be a short exact sequence of B-modules. Then it is also an exact sequence of A-modules. As F is flat, we get an exact sequence of A-modules

$$0 \to F \otimes_A N' \to F \otimes_A N \to F \otimes_A N'' \to 0.$$

To end the proof, we remark that for every B-module P we have an obvious isomorphism $F \otimes_A P \simeq G \otimes_B P$. □

Lemma 3.93 — *Let A be a ring and $S \subset A$ a multiplicative subset. Then $S^{-1}A$ is a flat A-algebra.*

Proof. Indeed for an A-module M we have $S^{-1}A \otimes_A M = S^{-1}M$. Moreover, $S^{-1}(-)$ takes short exact sequences to short exact sequences.

□

Lemma 3.94 — *Let A be a ring and M be an A-module. Then M is flat over A if and only if for any (maximal) prime $\mathfrak{p} \subset A$ $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module.*

Lemma 3.95 — *Let B be an A-algebra. Then B is a flat algebra if and only if for any prime $\mathfrak{q} \subset B$, $B_{\mathfrak{q}}$ is a flat $A_{\mathfrak{p}}$-algebra (with $\mathfrak{p}$ the inverse image of $\mathfrak{q}$).*

Proposition 3.96 — *Let A be a ring and F be an A-module. The following conditions are equivalent:*

1. *F is flat,*
2. *For every inclusion of A-modules $M' \subset M$ the morphism $F \otimes_A M' \to F \otimes_A M$ is injective.*
3. *For every ideal $I \subset A$ the morphism $F \otimes_A I \to F \otimes_A A \simeq F$ is injective.*

Proof. The equivalence between (1) and (2) follows from the fact that $F \otimes_A -$ is exact on the right, i.e., if $M \to M''$ is surjective then $F \otimes_A M \to F \otimes_A M''$ is surjective. (And this holds without any assumption on F.)

It remains to show that (3) implies (2). Let M be an A-module. We will say that F is M-flat if for any submodule $N \subset M$ the morphism $F \otimes_A N \to F \otimes_A M$ is injective. We know that F is A-flat and we want to show that F is M flat for any A-module. We do this in three steps:

Step 1: If F is M_1-flat and M_2-flat then F is $(M_1 \oplus M_2)$-flat.

Indeed, let $M = M_1 \oplus M_2$. For $N \subset M$, we let $N_1 = M_1 \cap N$ and N_2 be the image of N under the projection $M_1 \oplus M_2 \to M_2$. Thus, we have a short exact sequence

$$0 \to N_1 \to N \to N_2 \to 0.$$

We now have a commutative diagram

$$\begin{array}{ccccccccc} & & F \otimes_A N_1 & \longrightarrow & F \otimes_A N & \longrightarrow & F \otimes_A N_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_2} & & \\ 0 & \longrightarrow & F \otimes_A M_1 & \longrightarrow & F \otimes_A M & \longrightarrow & F \otimes_A M_2 & \longrightarrow & 0 \end{array}$$

We know that a_1 and a_2 are injective. It follows that the first line is also a short exact sequence. The claim follows now by the snake lemma.

Step 2: For any set I (possibly infinite), F is $A^{(I)}$-flat.

Indeed, let $N \subset A^{(I)}$ be a submodule. For $J \subset I$ a finite subset, we write $N_J = N \cap A^J$. Then the inclusion $N \subset A^{(I)}$ is the inductive limit of the inclusions $N_J \subset A^J$. Moreover, $F \otimes_A -$ commutes with filtered inductive limits.

Step 3: If F is M-flat and M'' is a quotient of M then F is M''-flat. Indeed, let N'' be a submodule of M'' and let N be the inverse image of N'' in M. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

Applying $F \otimes_A -$ we get

$$\begin{array}{ccccccccc} & & F \otimes_A M' & \longrightarrow & F \otimes_A N & \longrightarrow & F \otimes_A N'' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F \otimes_A M' & \longrightarrow & F \otimes_A M & \longrightarrow & F \otimes_A M'' & \longrightarrow & 0 \end{array}$$

The first horizontal line is then clearly a short exact sequence. We may use the snake lemma to conclude. □

Corollary 3.97 — *Let A be a d.v.r. with uniformizing π. Let F be an A-module. Then the following conditions are equivalent*

(1) *F is flat,*
(2) *F has no torsion, i.e., the submodule $\{m \in F; \pi.m = 0\}$ is zero.*

Proof. Follows immediately from Proposition ?? as $\{m \in F; \pi.m = 0\}$ is isomorphic to the kernel of $F \otimes_A (\pi) \to F \otimes_A A = F$. □

Definition 3.98 — ***1-*** *Let X be a scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{M}$ is flat if for every $x \in X$ the $\mathcal{O}_{X,x}$-module $\mathcal{M}_x$ is flat.*

2- *Let $f : Y \to X$ be a morphism of schemes. We say that f is a flat morphism if for every $y \in Y$, $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-algebra.*

Example 3.99 — Let k be a field. Any k-scheme $f : X \to \mathsf{Spec}(k)$ is flat.

Flatness of quasi-coherent modules and morphisms is a local notion. We also have:

Lemma 3.100 — ***1-*** *Let X be an affine scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{M}$ is flat if and only if $\mathcal{M}(|X|)$ is a flat $\mathcal{O}_X(|X|)$-module.*

2- *Let $f : Y \to X$ be a morphism between affine schemes. Then f is flat if and only if $\mathcal{O}_Y(|Y|)$ is a flat $\mathcal{O}_X(|X|)$-algebra.*

Corollary 3.101 — *Flat morphisms are preserved by base change.*

The main theorem of this paragraph is the following.

Theorem 3.102 — *Let $f : Y \to X$ be a morphism of finite type with X noetherian. If f is flat then it is universally open.*

Proof. We may assume that $Y = \mathsf{Spec}(B)$ and $X = \mathsf{Spec}(A)$ are affine schemes. It sufficed to show that $f(Y)$ is an open subset of X. We apply Proposition 3.80 to the constructible subset $X \backslash f(Y)$. Let R be a d.v.r. and $a : \mathsf{Spec}(R) \to X$ a morphism. We assume that $a(\eta) \notin f(Y)$ and we want to show that $a(o) \notin f(Y)$.

As $a(\eta) \notin f(Y)$ we deduce that the scheme $Y \times_X \eta$ is empty, i.e., $B \otimes_A \mathrm{Frac}(R) = 0$. Let $P = B \otimes_A R$. This is an R-algebra of finite type such that $P \otimes_R \mathrm{Frac}(R) = \pi^{-1}P = 0$ (where π is a uniformizing element of R). It follows that for some $n \in \mathbb{N}$ we have $\pi^n.1_P = 0$. This shows that P is a torsion R-module. But, we know that P is a flat R-module. This forces that $P = 0$. Hence, $Y \times_X o = \emptyset$, i.e., $o \notin f(Y)$. This completes the proof of the theorem. □

4. Dimension, Regularity and Smoothness

4.1. Combinatorial dimension of topological spaces.

Let X be a set. A chain in X is a strictly decreasing sequence

$$S_0 \supsetneq S_1 \supsetneq \cdots \supsetneq S_d$$

of subsets of X. The integer d is called the length of the chain.

Definition 4.1 — ***1-*** *Let X be a topological space. The* combinatorial dimension *(or simply* dimension*) of X is the supremum (in $\mathbb{N}\coprod\{-\infty,\infty\}$) of the lengths of chains of irreducible (hence non-empty) closed subsets of X:*

$$Z_0 \supsetneq \cdots \supsetneq Z_n.$$

We denote $\dim(X)$ *the dimension of X.*

If $x \in X$ we set $\dim_x(X)$ to be the infimum (in $\mathbb{N}\coprod\{\infty\}$) of the $\dim(U)$ where U runs over the open neighborhoods of x. This number is called the local dimension of X at x.

2- *Let $Y \subset X$ be a closed subset. Assume first that Y is irreducible. The* codimension *of Y (in X) is the supremum (in $\mathbb{N}\coprod\{\infty\}$) of the lengths of chains of irreducible closed subsets*

$$T_0 \supsetneq T_1 \supseteq \cdots \supsetneq T_n \supset Y$$

where T_i are irreducible closed subsets. We denote by $\operatorname{codim}_X(Y)$ *the codimension of Y in X.*

In the general case (where Y is not necessary irreducible), the codimension of Y is the infimum (in $\mathbb{N}\coprod\{\infty\}$) of $\operatorname{codim}_X(Z)$ where Z is an irreducible closed subset of Y.

Remark 4.2 — The combinatorial dimension of $\mathbb{R}^n$ is zero. By definition $\dim(\emptyset) = -\infty$ and $\operatorname{codim}_X(\emptyset) = \infty$. One can construct topological spaces having infinite dimension.

Here are some easy properties:

Lemma 4.3 — *Let X be a topological space.*

1. *If $Y \subset X$ is a subset, then $\dim(Y) \leq \dim(X)$. If Y is closed we have the more precise equality $\dim(X) \geq \dim(Y) + \operatorname{codim}_X(Y)$.*
2. $\dim(X) = \sup_{x\in X}\dim_x(X) = \sup_{\alpha\in I}\dim(X_\alpha)$ *where $\{X_\alpha, \alpha \in I\}$ is the set of irreducible components of X.*

Proof. For the first statement of (1) we take a chain of irreducible closed subsets of Y:

$$T_0 \supsetneq \cdots \supsetneq T_r.$$

Then the closure $\overline{T_i}$ of T_i is an irreducible closed subset. Moreover, as T_i is closed in Y we have $\overline{T_i} \cap Y = T_i$. This shows that

$$\overline{T_0} \supsetneq \cdots \supsetneq \overline{T_r}$$

is a chain of irreducible closed subsets of X. This proves the inequality $\dim(Y) \leq \dim(X)$.

Now, assume that Y is closed. To show the inequality $\dim(X) \geq \dim(Y) + \operatorname{codim}_X(Y)$ we may assume that $\dim(X) < +\infty$. Let $Z_0 \supsetneq \cdots \supsetneq Z_r$ be a chain of irreducible subsets of Y of maximal length, i.e., $r = \dim(Y)$. Also, let

$$T_0 \supsetneq \cdots \supsetneq T_s = Z_0$$

be a chain of irreducible closed subsets of maximal length (among those who ends at Z_0), i.e., $s = \operatorname{codim}_X(Z_0)$. Then

$$\dim(X) \geq r + s = \dim(Y) + \operatorname{codim}_X(Z_r) \geq \dim(Y) + \operatorname{codim}_X(Y).$$

The equality $\dim(X) = \sup_\alpha \dim(X_\alpha)$ is obvious as every chain of irreducible subsets is contained in at least one of the X_α.

The inequality $\dim(X) \geq \sup_{x \in X} \dim_x(X)$ follows from the fact that $\dim(X) \geq \dim(U)$ for any open subset U of X. Conversely, let

$$Z_0 \supsetneq \cdots \supsetneq Z_n$$

be a chain of irreducible closed subsets in X. Fix $x \in Z_n$. For every open neighborhood U of x, $U \cap Z_i$ is dense in Z_i. This implies that $Z_i \cap U \supsetneq Z_{i+1} \cap U$. We thus have a chain of irreducible closed subsets of U of length n. □

Definition 4.4 — *Let X be a topological space. We say that X is* of pure dimension $n \in \mathbb{N} \coprod \{\infty\}$ *if every irreducible component of X has dimension n.*

Definition 4.5 — *Let X be a scheme. The* dimension *of X is the dimension of the topological space $|X|$. If $Z \subset X$ is a closed subscheme, we call the codimension of Z the codimension of $|Z|$.*

Proposition 4.6 — *Let $f : Y \to X$ be a finite morphism. Then* $\dim(Y) \leq \dim(X)$. *If moreover, f is dominant, we have* $\dim(Y) = \dim(X)$.

Proof. This follows immediately from the fact that the fibers of f are discrete topological spaces. Indeed, let $Y_0 \supsetneq \cdots \supsetneq Y_r$ be a chain of cosed irreducible subsets of Y. Then $f(Y_i) \neq f(Y_{i+1})$. Indeed, assume that $f(Y_i) = f(Y_{i+1})$. Let x be the generic point of this subset. As $f^{-1}(x)$ is discrete, we should have $(Y_i)_x = (Y_{i+1})_x$. But this implies that $Y_i = Y_{i+1}$.

Now assume that f is dominant. Then f is surjective and it is easy to get the converse inequality. □

4.2. Dimension of rings and height of ideals.

Definition 4.7 — ***1-*** *Let A be a ring. The* Krull dimension *(or simply* dimension*) of A is the dimension of* $\mathsf{Spec}(A)$. *Thus it is the supremum of the length of strictly increasing chain of prime ideals*

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

We denote $\dim(A)$ *the dimension of A.*

2- *Let $I \subset A$ be an ideal. The* height *(or sometimes the* codimension*) of I is the codimension of $\mathcal{Z}(I)$ in* $\mathsf{Spec}(A)$. *It is denoted by* $\operatorname{ht}(I)$.

Lemma 4.8 — *Let A be a ring and $\mathfrak{p} \subset A$ a prime ideal. Then* $\operatorname{ht}(\mathfrak{p}) = \dim(A_\mathfrak{p})$.

Proof. Given a strictly increasing chain of prime ideals

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n \subset \mathfrak{p}$$

we deduce an increasing chain of prime ideals in the local ring $A_\mathfrak{p}$

$$\mathfrak{q}_0 A_\mathfrak{p} \subset \cdots \subset \mathfrak{q}_n A_\mathfrak{p} \subset \mathfrak{p} A_\mathfrak{p}.$$

This chain is strictly increasing as $\mathfrak{q}_i$ is the inverse image of $\mathfrak{q}_i A_\mathfrak{p}$ by the obvious morphism $A \to A_\mathfrak{p}$. This shows that $\mathrm{ht}(\mathfrak{p}) \leq \dim(A_\mathfrak{p})$. The inequality in the other direction is proven in the same way. □

Before stating the main theorem of this paragraph we recall some facts about artin rings. Recall that an artin ring A is a noetherian ring where every prime ideal is maximal (and thus also minimal). As there are only finitely many minimal prime ideals in a noetherian ring, A has only finitely many prime ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. Moreover, the obvious map $A \to \prod_{i=1}^r A_{\mathfrak{m}_r}$ is an isomorphism. Let M be a finite type A-module. We say that M is simple if it is not zero and has no non-zero strict sub-module. On can easily show that simple A-modules are of the form $A/\mathfrak{m}_i$ for $1 \leq i \leq r$. If M is a finite type A-module, we can find an increasing filtration

$$0 \subsetneq M_0 \subsetneq M_1 \cdots \subsetneq M_n = M$$

such that M_i/M_{i-1} is simple for $1 \leq i \leq n$. The integer n is independent of the filtration and is called the length of M. It will be denoted by $\mathrm{lg}(M)$.

Theorem 4.9 — *Let A be a noetherian ring and $\mathfrak{a} \subset A$ an ideal generated by d elements. Let $\mathfrak{p}$ be a minimal prime ideal containing $\mathfrak{a}$. Then $\mathrm{ht}(\mathfrak{p}) \leq d$. In particular $\mathrm{ht}(\mathfrak{a}) \leq d$.*

Proof. Assume that $\mathfrak{a} = (a_1, \ldots, a_d)$. We may replace A and $\mathfrak{a}$ by $A_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p}$. Thus we may assume that A is local with maximal ideal $\mathfrak{m} = \sqrt{\mathfrak{a}}$. We will argue on d, the number of generators of $\mathfrak{a}$.

Step 1: The case $d = 1$. Here we assume that A is a noetherian local ring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} = \sqrt{(a)}$ for some $a \in A$. We need to show that $\mathfrak{m}$ is of height 1. Assume by contradiction that we may find a strictly increasing chain of ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{m}.$$

Replacing A, $\mathfrak{p}_1$ and $\mathfrak{m}$ by $A/\mathfrak{p}_0$, $\mathfrak{p}_1/\mathfrak{p}_0$ and $\mathfrak{m}/\mathfrak{p}_0$ we may assume that A is integral and that we have a non-zero prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$.

By assumption, $\mathfrak{m}$ is the minimal prime ideal containing (a). Thus, $a \notin \mathfrak{p}$. Moreover, we fix $b \in \mathfrak{p}$ a non-zero element. We consider the sequence on ideals $(a^n, b) \subset A$. As $A/(a^n)$ is an artinian local ring, we can speak of $\mathrm{lg}(A/(a^n, b))$. We have

$$\mathrm{lg}(A/(a^n, b)) = \mathrm{lg}(A/(a^n)) - \mathrm{lg}((a^n, b)/(a^n)) = \mathrm{lg}(A/(a^n)) - \mathrm{lg}((b)/(b) \cap (a^n)).$$

We denote by $b^{-1}[(b) \cap (a^n)]$ the ideal of A that consists of elements c such that $bc \in (b) \cap (a^n)$. As b is a non-zero divisor, multiplication by b yields an isomorphism

$$A/(b^{-1}[(b) \cap (a^n)] \simeq (b)/((b) \cap (a^n)).$$

We thus get

$$\mathrm{lg}(A/(a^n, b)) = \mathrm{lg}(A/(a^n)) - \mathrm{lg}(A/b^{-1}[(b) \cap (a^n)]) = \mathrm{lg}\left(\frac{b^{-1}[(b) \cap (a^n)]}{(a^n)}\right)$$

Now we have isomorphisms of A-modules

$$\frac{b^{-1}[(b)\cap(a^n)]}{(a^n)} \simeq \frac{(b)\cap(a^n)}{(ba^n)} \simeq \frac{a^{-n}[(b)\cap(a^n)]}{(b)}.$$

As A is noetherian the increasing sequence of ideals

$$a^{-n}[(b)\cap(a^n)] \subset a^{-n-1}[(b)\cap(a^{n+1})] \subset \dots$$

stabilize. This show that the sequence of numbers $\lg(A/(b,a^n))$ stabilize. We thus get an integer s such that $(b,a^s)=(b,a^{s+1})$. This means that we may write $a^s = ub + va^{s+1}$. But then we get $ub = a^s(1+va)$. As $(1+va)$ is invertible, we see that $a \in \mathfrak{p}$. This is a contradiction.

Step 2: The general case. We now go back to the general. We assume the result true for $d-1$. We argue by contradiction. Hence, we may find a strictly increasing sequence of prime ideals

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d \subsetneq \mathfrak{m}$$

Replacing A by $A/\mathfrak{p}_0$, we may assume that A is integral and $\mathfrak{p}_0=(0)$. Replacing $\mathfrak{p}_d$ by a larger prime ideal, we may assume that there is no prime ideal $\mathfrak{q}$ such that $\mathfrak{p}_d \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$.

As $\mathfrak{m}$ is the minimal prime ideal containing $(a_1,\dots,a_d)$, we may assume that $a_d \notin \mathfrak{p}_d$. By our choice of $\mathfrak{p}_d$, we know that $\mathfrak{m}$ is the minimal ideal containing $(\mathfrak{p}_d + (a_d))$. This implies that $a_i \in \sqrt{(\mathfrak{p}_d + (a_d))}$ for all $1 \le i \le d-1$. Thus, we may find an integer N such that $a_i^N = b_i + u_i a_d$ with $b_i \in \mathfrak{p}_d$ and $1 \le i \le d-1$.

Consider now the ideal $\mathfrak{b} = (b_1,\dots,b_{d-1})$. By the induction hypothesis, $\mathfrak{p}_d$ is not a minimal prime ideal containing $\mathfrak{b}$. Hence, we may find a prime ideal $\mathfrak{q}$ such that $\mathfrak{b} \subset \mathfrak{q} \subsetneq \mathfrak{p}_d$. Let $B = A/\mathfrak{q}$. Denote by a_i' the images of a_i in B. We see that $a_i' \in \sqrt{(a_d')}$ so that $\mathfrak{m}' = \mathfrak{m}/\mathfrak{q}$ is the minimal prime ideal containing a_d'. By the case $d=1$, we know that $\mathrm{ht}(\mathfrak{m}') \le 1$. This is a contradiction as we have a strictly increasing sequence

$$(0) \subsetneq \mathfrak{p}_d/\mathfrak{q} \subsetneq \mathfrak{m}'.$$

This finishes the proof of the theorem. □

Proposition 4.10 — *Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal of height d. Then there exists $a_1,\dots,a_d \in \mathfrak{p}$ such that:*

(1) *$\mathfrak{p}$ is a minimal prime ideal containing $(a_1,\dots,a_d)$.*
(2) *For all $1 \le k \le d$, $\mathrm{ht}(a_1,\dots,a_k) = k$.*

Proof. First we prove the case $d=1$. We clearly may assume that A is reduced. The condition $\mathrm{ht}(a_1)$ is then equivalent to the property that a is a non-zero divisor (use Lemma 3.36). Thus, we must find a non-zero divisor of $\mathfrak{p}$. This is possible as $\mathfrak{p}$ is not minimal.

We argue by induction on d. We can find $\mathfrak{q} \subset \mathfrak{p}$ such that $\mathrm{ht}(q)+1 = \mathrm{ht}(p)$. Let $a_1,\dots,a_{d-1} \in \mathfrak{q}$ satisfying to the conditions of the proposition. Let $B = A/(a_1,\dots,a_{d-1})$ and denote by $\mathfrak{q}_0$ and $\mathfrak{p}_0$ the images of $\mathfrak{q}$ and $\mathfrak{p}$ in B. Then B is a reduced ring, $\mathfrak{q}_0$ is a minimal prime ideal of B and $\mathfrak{p}_0$ has height 1. Thus, by the case $d=1$, we may find $b \in \mathfrak{p}_0$ a non-zero divisor. An antecedent $a_d \in A$ of b satisfies the wanted properties. □

4.3. **Dimension of finite type k-algebras and finite type k-schemes.**

Theorem 4.11 — *Let A be a noetherian ring. Then* $\dim(A[t]) = \dim(A) + 1$.

We need some lemmas.

Lemma 4.12 — *Let $\mathfrak{q} \subsetneq \mathfrak{p}$ be prime ideals of $A[t]$. Assume that $\mathfrak{q} \cap A = \mathfrak{p} \cap A$. Then $\mathfrak{q} = (\mathfrak{q} \cap A)A[t]$.*

Proof. Replacing A by $A/(\mathfrak{q} \cap A)$, we may assume that $\mathfrak{q} \cap A = \mathfrak{p} \cap A = 0$. Our goal is to show that $\mathfrak{q} = 0$. It follows that the multiplicative subset $S = A \backslash \{0\} \subset A[t]$ is disjoint from $\mathfrak{q}$ and $\mathfrak{p}$. In particular, it suffices to show that $S^{-1}\mathfrak{q} = 0$ while we know that $S^{-1}\mathfrak{q}$ is not a maximal ideal as it is strictly contained in $S^{-1}\mathfrak{p}$. But $S^{-1}A[t] = \mathrm{Frac}(A)[t]$ is a principal ideal domain, and hence of dimension 1. □

The following lemma implies immediately Theorem 4.11.

Lemma 4.13 — *Let A be a noetherian ring and $\mathfrak{p}$ a prime ideal of $A[t]$. Then the following holds:*

(1) *If $\mathfrak{p} = (A \cap \mathfrak{p})A[t]$, then* $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p} \cap A)$.
(2) *If $\mathfrak{p} \neq (A \cap \mathfrak{p})A[t]$, then* $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p} \cap A) + 1$.

Proof. Let $\mathfrak{q} = A \cap \mathfrak{p}$ and denote $d = \mathrm{ht}(\mathfrak{q})$.

To show (1), we use Proposition 4.10 to find an ideal $(a_1, \ldots, a_d) \subset A$ such that $\mathfrak{p} \cap A$ is a minimal prime ideal containing it. But then $\mathfrak{q}[t]$ is a minimal prime ideal of $A[t]$ containing $(a_1, \ldots, a_d)$. We thus get $\mathrm{ht}(\mathfrak{p}) \leq d$.

We now turn to (2). Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset \mathfrak{p}$ be a chain of prime ideals in $A[t]$. Let Denote by $\mathfrak{q}_i = \mathfrak{p}_i \cap A$. If $\mathfrak{q}_i \subsetneq \mathfrak{q}_{i+1}$ we have nothing to show. Otherwise, let s be the maximal i such that $\mathfrak{q}_s = \mathfrak{q}_{s+1}$. By the Lemma above and (1), we get that $\mathrm{ht}(\mathfrak{q}_s) = \mathrm{ht}(\mathfrak{p}_s)$. This proves our claim. □

Corollary 4.14 — *Let k be a field. Then* $\dim(k[t_1, \ldots, t_n]) = n$.

Lemma 4.15 — *Let k be a field and X a finite type k-scheme. Let $U \subset X$ be a dense open subset. Then* $\dim(X) = \dim(U)$.

Proof. Let $(X_\alpha)_\alpha$ be the irreducible components of X. As $\dim(X) = \sup_\alpha \dim(X_\alpha)$ and $\dim(U) = \sup_\alpha \dim(X_\alpha \cap U)$, it suffices to show that $\dim(X_\alpha) = \dim(X_\alpha \cap U)$. Thus we may assume that X is irreducible (and hence integral as $\dim(X) = \dim(X_\alpha)$).

Similarily, let $(X_i)_{i \in I}$ be an open cover of X. As $\dim(X) = \sup_i \dim(X_i)$ and $\dim(U) = \sup_i \dim(U \cap X_i)$, it suffices to show that $\dim(X_i) = \dim(U \cap X_i)$. We are thus reduced to the case where $X = \mathsf{Spec}(A)$ is affine with A an integral domain.

We may find $f \in A$ such that $\emptyset \neq \mathrm{D}(f) \subset U$. As $\dim(\mathrm{D}(f)) \leq \dim(U) \leq \dim(X)$, we may assume that $U = \mathrm{D}(f)$.

We argue by induction on $d = \dim(X)$. We may assume that $d \geq 1$. Let $(0) = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ be a maximal chain of prime ideals in X. If $f \notin \mathfrak{p}_d$, then we have a chain $(0) = (\mathfrak{p}_0)_f \subsetneq \cdots \subsetneq (\mathfrak{p}_d)_f$ of prime ideals in A_f and we are done. So we may assume that $f \in \mathfrak{p}_d$. Let $s \in [\![1, d]\!]$ be the integer such that $f \in \mathfrak{p}_s \backslash \mathfrak{p}_{s-1}$. We will show that we can find a prime ideal $\mathfrak{p}_{s-1} \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_{s+1}$ (where, for convenience we set $\mathfrak{p}_{d+1} = A$) such that $f \notin \mathfrak{q}$. Thus replacing $\mathfrak{p}_s$ by $\mathfrak{q}$ we get a chain of length d for which the number of prime ideals that do not contain f is strictly improved. This will clearly imply the lemma.

To find the ideal $\mathfrak{q}$, we distinguish two cases. First, let's assume that $s = d$. In this case, we need to find a maximal ideal containing $\mathfrak{p}_{d-1}$ but not f. So we assume by contradiction that every maximal ideal of A containing $\mathfrak{p}_{d-1}$ also contains f. This means that the maximal ideals of $A/\mathfrak{p}_{d-1}$ are in 1-to-1 correspondence with the maximal ideals of $A/(f) + \mathfrak{p}_{d-1}$. As the Krull dimension of the integral domain $A/\mathfrak{p}_{d-1}$ is 1, we see that every prime ideal of $A/(f) + \mathfrak{p}_{d-1}$ is actually a maximal ideal. This implies that $A/(f)+\mathfrak{p}_{d-1}$ is an artinian ring, and in particular have only finitely many prime ideals. We thus have shown that the finitely generated k-algebra $A/\mathfrak{p}_{d-1}$ has only finitely many maximal ideals. By Lemma 3.52 we see that $A/\mathfrak{p}_{d-1}$ is also an artinian ring. This is a contradiction as the Krull dimension of $A/\mathfrak{p}_{d-1}$ is 1.

Now we assume that $s < d$ so that $\mathfrak{p}_{s+1} \subsetneq A$. Assume also by contradiction that for every prime ideal $\mathfrak{q}$ such that $\mathfrak{p}_{s-1} \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_{s+1}$ contains f. As $\mathfrak{q}$ must be a minimal prime ideal containing $\mathfrak{p}_{s-1} + (f)$ we deduce that there is finitely many prime ideal strictly betweem $\mathfrak{p}_{s-1}$ and $\mathfrak{p}_{s+1}$. This is a contradiction by the Lemma below. □

LEMMA 4.16 — *Let A be a noetherian local ring of dimension $d \geq 2$. Then there is infinitely many prime ideals in A of height $d-1$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of height $d-2$ and such that $A/\mathfrak{p}$ is of dimension 2. We may replace A by $A/\mathfrak{p}$. Thus, we may assume that $\dim(A) = 2$ and that A is an integral domain.

Let $\mathfrak{m}$ be the maximal ideal of A. We argue by contradiction assuming that A has finitely many prime ideals. Let $(a_1, a_2) \subset A$ be an ideal such that $\mathfrak{m}$ is the unique minimal prime ideal containing (a_1, a_2) (This is possible by Proposition 4.10.

Let $S(n)$ be the set of minimal prime ideals containing $(a_1 + a_2^n)$. Then $S(n) \cap S(m) = \emptyset$ if $0 < m < n$. Indeed, let $\mathfrak{p} \in S(n) \cap S(m)$. Then $\mathfrak{p}$ contains $a_2^m(1-a_2^{n-m})$. As $1 - a_2^{n-m}$ is invertible in A, we deduce that $\mathfrak{p}$ contains a_2^m and hence a_2. But then $\mathfrak{p}$ contains also a_1. This means that $\mathfrak{p} = \mathfrak{m}$ which is a contradiction as $\mathfrak{m}$ is of height 2.

As $\mathsf{Spec}(A)$ is a finite set, we see that $S(n) = \emptyset$ for n large enough. This is impossible. □

THEOREM 4.17 — *Let k be a field and X a finite type k-scheme. Let $\{\eta_i,\ \alpha \in I\}$ be the set of generic points in X. Then $\dim(X) = \sup_{i \in I} \operatorname{degtr}(k(\eta_i)/k)$ where $\operatorname{degtr}(K/k)$ is the transcendence degree of a field extension K of k.*

Proof. We may replace X by one of its irreducible components. So we may assume that X is an integral scheme. Using Lemma 4.15 we may assume that $X = \mathsf{Spec}(A)$ is affine with A an integral domain. We wish to show that

$$\dim(A) = \operatorname{degtr}(\operatorname{Frac}(A)/k).$$

Let $n = \operatorname{degtr}(\operatorname{Frac}(A/k))$. We may find a morphism of k-extensions $k(t_1, \ldots, t_n) \to \operatorname{Frac}(A)$ such that every element of $\operatorname{Frac}(A)$ becomes algebraic over $k(t_1, \ldots, t_n)$. As $\operatorname{Frac}(A)/k$ is a finitely generated field extension, we deduce that $\operatorname{Frac}(A)/k(t_1, \ldots, t_n)$ is also finitely generated and hence a finite extension.

Let $f \in A\backslash\{0\}$ be such that the image of t_i in $\mathrm{Frac}(A)$ is in A_f. Replacing A by A_f (which is possible as $\dim(A) = \dim(A_f)$ by Lemma 4.15) we may actually assume that the images of t_i are in A. Thus, we have a morphism of k-algebras

$$k[t_1, \ldots, t_n] \to A.$$

Let $b_1, \ldots, b_r$ be generators of the k-algebra A. As every element of A is algebraic over $k(t_1, \ldots, t_n)$, we may find polynomials $P_i \in k(t_1, \ldots, t_n)[s]$ unitary with $P_i(b_i) = 0$. Let $Q \in k[t_1, \ldots, t_n]$ be a common denominator of the coefficients of the P_i and let $g \in A$ be its image. Then $\mathsf{Spec}(A_g) \to \mathsf{Spec}(k[t_1, \ldots, t_n]_Q)$ is a finite map and dominant map. This implies that $\dim(A_g) = \dim(k[t_1, \ldots, t_n]_Q)$. Applying Lemma 4.15, we get that $\dim(A) = \dim(k[t_1, \ldots, t_n]) = n$. □

4.4. **Sets of Parameter in local rings.**

Definition 4.18 — *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and Krull dimension d. A set $\{a_1, \ldots, a_d\}$ of elements in $\mathfrak{m}$ is called a* set of parameters *if $\mathfrak{m}$ is the only minimal prime ideal containing $(a_1, \ldots, a_n)$. Equivalently, $A/(a_1, \ldots, a_d)$ is a artinian ring.*

By Proposition 4.10 a noetherian local ring always have a set of parameters. Indeed, if $(a_1, \ldots, a_d)$ is such that $\mathrm{ht}(a_1, \ldots, a_d) = d$, then no prime ideal other than the maximal one can contain $(a_1, \ldots, a_n)$.

Lemma 4.19 — *Let A be a local noetherian ring. Let $a \in \mathfrak{m}$ be such that its class in A_{red} is a non-zero divisor. Then $\dim(A/(a)) = \dim(A) - 1$.*

Proof. We can easily reduce to the case where A is integral. We know that $\dim(A/(a)) \leq \dim(A) - 1$. We need to show the opposite inequality. Let

$$(0) \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_{d-1} \subsetneq \mathfrak{p}_d = \mathfrak{m}$$

be a maximal chain of prime ideals. If $a \in \mathfrak{p}_1$ we have nothing to prove. So we may assume that $a \notin \mathfrak{p}_1$. Let $s \in [\![1, d-1]\!]$ be such that $a \in \mathfrak{p}_{s+1}\backslash\mathfrak{p}_s$. We argue by descending induction on s. Let $B = (A/\mathfrak{p}_{s-1})_{\mathfrak{p}_{s+1}}$. This is a noetherian integral local domain of dimension 2. Let $\bar{a}$ be the class of a in B. Then $\bar{a} \neq 0$. Moreover, if $\mathfrak{q} \subset B$ is a minimal prime ideal containing a, we have $\mathrm{ht}(\mathfrak{q}) = 1$ and $\mathfrak{q}$ is not the maximal ideal. Thus, if we denote by $\mathfrak{q}'$ the inverse image of $\mathfrak{q}$ is A, we have:

$$\mathfrak{p}_{s-1} \subsetneq \mathfrak{q}' \subsetneq \mathfrak{p}_{s+1}$$

and moreover, $a \in \mathfrak{q}'$. This finishes the proof of the lemma. □

Corollary 4.20 — *Let A be a noetherian local ring of dimension d. Let $a_1, \ldots, a_r$ be a sequence of elements in $\mathfrak{m}$ such that:*

1. *The class of a_1 is a non-zero divisor in A_{red},*
2. *For every $i \in [\![2, r]\!]$, the class of a_i is a non-zero divisor in $(A/(a_1, \ldots, a_{i-1}))_{red}$.*

Then, we may find elements $a_{r+1}, \ldots, a_d$ such that $\{a_1, \ldots, a_d\}$ is a set of parameters in A. In particular, if $r = d$, we already have a sequence of parameters.

Proof. Indeed, by Lemma 4.19 we know that $\dim(A/(a_1, \ldots, a_r)) = d - r$. Thus, we may find $b_{r+1}, \ldots, b_d \in A/(a_1, \ldots, a_r)$ a set of parameters. If $a_{r+1}, \ldots, a_d \in A$ are lifts to the b_i, we see that $A/(a_1, \ldots, a_d)$ is an artinian ring. □

Proposition 4.21 — *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $\{a_1, \ldots, a_d\}$ be a set of parameters of A. Then $\dim(A/(a_1, \ldots, a_r)) = d - r$ and the residues $\{\bar{a}_{r+1}, \ldots, \bar{a}_d\}$ is a set of parameters for $A/(a_1 \ldots, a_r)$.*

Proof. It is clear that $\mathfrak{m}/(a_1, \ldots, a_r)$ is the minimal prime ideal containing $(\bar{a}_{r+1}, \ldots, \bar{a}_d)$. This implies that $\dim(A/(a_1, \ldots, a_r)) \leq d - r$. If the inequality was strict, then we would be able to find elements $b_1, \ldots, b_s \in \mathfrak{m}$ with $s < d - r$ such that

$$A/(a_1, \ldots, a_r, b_1, \ldots, b_s)$$

is artinian. This would implies that $\dim(A) < d$. This is impossible. □

Definition 4.22 — *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. An ideal $\mathfrak{a} \subset A$ is called a* parameter ideal *if it can be generated by a set of parameters in A.*

Let A be a local ring with maximal ideal $\mathfrak{m}$. Let $\mathfrak{a}$ be an ideal in A. We may form a $\kappa(\mathfrak{m})$-algebra $\oplus_{n \in \mathbb{N}} \mathfrak{a}^n/\mathfrak{m}\mathfrak{a}^n$. Moreover, given elements $a_1, \ldots, a_n \in \mathfrak{a}$, we deduce a morphism of graded rings

$$\kappa(\mathfrak{m})[t_1, \ldots, t_n] \to \bigoplus_{n \in \mathbb{N}} \mathfrak{a}^n/\mathfrak{m}\mathfrak{a}^n$$

by sending t_i to $a_i + \mathfrak{a}\mathfrak{m}$.

Theorem 4.23 — *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $\mathfrak{a} \subset A$ be a parameter ideal generated by a set of parameters $\{a_1, \ldots, a_d\}$. Then the morphism*

$$\omega : k[t_1, \ldots, t_n] \to \bigoplus_{n \in \mathbb{N}} \mathfrak{a}^n/\mathfrak{m}\mathfrak{a}^n \tag{11}$$

sending t_i to the class of a_i is an isomorphism.

Proof. The morphism (11) is clearly surjective. We need to prove injectivity. Let $\mathfrak{k} \subset k[t_1, \ldots, t_n]$ be the kernel. This is a graded ideal. Assume by contradiction that $\mathfrak{k}$ is not zero. Hence, we may find $P_1 \in \mathfrak{k}$ homogenous of non-zero degree. By Lemma ?? below, we may find $P_2, \ldots, P_n$ homogenous such that $k[t_1, \ldots, t_n]$ is a finite algebra over $k[P_1, \ldots, P_n]$. Replacing the P_i by convenient powers, we may assume that the P_i have the same degree, say p.

Now $\omega(P_1) = 0$ and we can write $\omega(P_i) = v_i + \mathfrak{m}\mathfrak{a}^p$. Let $\mathfrak{b} \subset A$ be the ideal generated by $v_2, \ldots, v_n$. We will show that $\mathfrak{a} \subset \sqrt{\mathfrak{b}}$. This will be a contradiction.

Now, recall that t_i is algebraic over $k[P_1, \ldots, P_n]$. Hence, we may find equations

$$t_i^N + Q_{i,N-1}(P_1, \ldots, P_n)t_i^{N-1} + \cdots + Q_{i,0}(P_1, \ldots, P_n) = 0.$$

As P_i are homogenous, we may also assume that $Q_{i,j}$ are homogenous. Applying ω to these equations yield

$$a_i^N + \sum_{j=0}^{N-1} Q_{i,j}(0, v_2, \ldots, v_n)a_i^j \in \mathfrak{m}\mathfrak{a}^N.$$

This gives that $a_i^N \in \mathfrak{m}\mathfrak{a}^N + \mathfrak{b}$. It follows that $\mathfrak{a}^{Nn} \subset \mathfrak{m}\mathfrak{a}^{Nn} + \mathfrak{b}$. By Nakayama's lemma applied to $A/\mathfrak{b}$ we deduce that $\mathfrak{a}^{Nn}(A/\mathfrak{b}) = 0$. This means that $\mathfrak{a}^{Nn} \subset \mathfrak{b}$ which is the desired contradiction. □

Lemma 4.24 — *Let $P_1 \in k[t_1, \dots, t_n]$ be a non-zero homogenous element. Then, there are $P_2, \dots, P_n$ homogenous such that $k[t_1, \dots, t_n]$ is a finite algebra over $k[P_1, \dots, P_n]$.*

Proof. We choose P_i so that its class in $(k[t_1, \dots, t_n]/(P_1, \dots, P_{i-1}))_{red}$ is not a zero divisor. We then have $\dim(k[t_1, \dots, t_n]/(P_1, \dots, P_n)) = 0$. This means that this ring $B = k[t_1, \dots, t_n]/(P_1, \dots, P_n)$ is a graded artinian ring. Moreover, all the prime ideals of B should be homogenous. In particular, we see that we have exactly one prime ideal, namely B_+. This means that B_+ is nilpotent. In particular, $B_N = 0$ for some N big enough.

Let $C \subset k[t_1, \dots, t_n]$ be the sub-$k[P_1, \dots, P_n]$-module generated by homogenous polynomials in $t_1, \dots, t_n$ of degree less or equal to N. We will show by induction that $k[t_1, \dots, t_n]_r \subset C$ by induction on r.

When $r \leq n$ there is nothing to prove. Let $Q \in k[t_1, \dots, t_n]_{r+1}$ with $r \geq N$. Then the class of Q is zero in B. This means that we may write $Q = \sum_{i=1}^n T_i P_i$ with T_i homogenous of degree $r + 1 - \deg(P_i)$. But, by the induction hypothesis, we know that $T_i \in C$. This complete the induction.

The lemma now follows as $k[P_1, \dots, P_n]$ is noetherian and we have just shown that $k[t_1, \dots, t_n]$ is a finite type module over $k[P_1, \dots, P_n]$. ☐

4.5. Local regular rings.

Definition 4.25 — *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. We say that A is* regular *if $\mathfrak{m}$ is a parameter ideal, i.e., it can be generated by* $\mathrm{ht}(\mathfrak{m})$ *elements.*

Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. For every $n \in \mathbb{N}$, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a $\kappa(\mathfrak{m})$-vector space. Moreover, $\mathrm{Gr}(A, \mathfrak{m}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finite type $\kappa(\mathfrak{m})$-algebra. We have the following.

Proposition 4.26 — *Let A be a noetherian local ring. Assume that A is regular. Then* $\mathrm{Gr}(A, \mathfrak{m})$ *is isomorphic to a polynomial $\kappa(\mathfrak{m})$-algebra.*

Proof. This follows immediately from Theorem 4.23. ☐

The converse of the previous proposition is also true but will not be used later.

Theorem 4.27 — *A local regular ring is an integral domain.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A. As A is regular, we have $\mathfrak{m} = (a_1, \dots, a_d)$ with d the dimension of A. Now, let u and v be two non-zero elements of A such that $uv = 0$ but $u \neq 0$ and $v \neq 0$. We want to derive a contradiction.

As A is noetherian, we know that $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$. Hence, we may find integers m and n such that $u \in \mathfrak{m}^m \backslash \mathfrak{m}^{m+1}$ and $v \in \mathfrak{m}^n \backslash \mathfrak{m}^{n+1}$. Then $u + \mathfrak{m}^{m+1}$ and $v + \mathfrak{m}^{n+1}$ are two non-zero elements in $\mathrm{Gr}(A, \mathfrak{m})$ and their product vanish. This is impossible as $\mathrm{Gr}(A, \mathfrak{m})$ is a polynomial $\kappa(\mathfrak{m})$-algebra and hence integral. ☐

Corollary 4.28 — *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ generated by $a_1, \dots, a_d$ where $d = \dim(A)$. Then $(a_1, \dots, a_r)$ is a prime ideal and $A/(a_1, \dots, a_r)$ is a regular ring.*

Remark 4.29 — It is known that a local ring is regular if and only if it is factorial. In particular, if A is a local regular ring and $\mathfrak{p} \subset A$ is a prime ideal, then

$A_\mathfrak{p}$ is also a regular local ring. However, the proof of these properties are based on the homological characterization of local regular rings as those local rings having finite global homological dimension.

4.6. **Kähler differentials.**

Definition 4.30 — *Let A be a ring and B an A-algebra. An A-*derivation *of B with values in a B-module M is an application*

$$d : B \to M$$

satisfying to the following rules:

(a) *d is A-linear, i.e., a morphism of A-modules.*
(b) *(Leibniz rule) $d(b_1 b_2) = b_1.d(b_2) + b_2.d(b_1)$ for b_1, $b_2 \in B$.*

The set of A-derivations from B to M is denoted by $\mathrm{Der}_A(B, M)$. It is naturally a B-module.

Theorem 4.31 — *There exists a universal A-derivation*

$$\mathrm{d}_{B/A} : B \to \Omega_{B/A}$$

in the following sense. For any other A-derivation, $d : B \to M$ there is a unique B-linear morphism $\Omega_{B/A} \to M$ making the following triangle

$$\begin{array}{ccc} B & \xrightarrow{\mathrm{d}_{B/A}} & \Omega_{B/A} \\ & {\scriptstyle d}\searrow & \downarrow \\ & & M \end{array}$$

commutative. In other words, the obvious morphism

$$\hom(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$$

is invertible.

Moreover, the image of $\mathrm{d}_{B/A}(B)$ generates the B-module $\Omega^0_{B/A}$ and if B is an A-algebra of finite type, then $\Omega_{B/A}$ is a finitely generated B-module.

Proof. Let F be the free B-module generated by the symbols $\underline{d}(b)$ for $b \in B$. Let $G \subset F$ be the sub-B-module generated by the following elements

$$\underline{d}(ab) - a\underline{d}(b) \ , \quad \underline{d}(bb') - b\underline{d}(b') - b'\underline{d}(b).$$

We set $\Omega_{B/A} = F/G$ and define $\mathrm{d}(b)$ to be the class of $\underline{d}(b)$ in the quotient. The universal property is easily checked. Also by construction, $\Omega_{B/A}$ is generated as a B-module by $\mathrm{d}(B)$.

Now assume that B is a finitely generated A-algebra. Let $b_1, \dots, b_n$ be generators. Then every element of B is a polynomial $P(b_1, \dots, b_n)$ where $P \in A[t_1, \dots, t_n]$. We see immediately that

$$\mathrm{d}(P(b_1, \dots, b_n)) = \sum_{i=1}^{n} \frac{\partial P}{\partial t_i}(b_1, \dots, b_n)\mathrm{d}(b_i).$$

This shows that $\Omega^0_{B/A}$ is generated by $\mathrm{d}(b_i)$. □

Example 4.32 — Let A be a ring. Then $\Omega_{A[t_1,\ldots,t_n]/A}$ is a free $A[t_1,\ldots,t_n]$-module of rank n generated by $\mathrm{d}t_1,\ldots,\mathrm{d}t_n$. Indeed, let $B = A[t_1,\ldots,t_n]$. We may define a derivation in the usual way

$$B \to \bigoplus_{i=1}^{n} B.\mathrm{d}t_i$$

sending a polynomial P to $\sum_{i=1}^{n} \frac{\partial P}{\partial t_i}\mathrm{d}t_i$. As this formula holds for any differential, we see that d is universal.

Proposition 4.33 — *Let $A \to B \to C$ be morphisms of rings. We have a canonical exact sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Proof. As $\mathrm{d} : C \to \Omega_{C/B}$ is an A-derivation, it gives rise to a morphism of B-modules $\Omega_{C/A} \to \Omega_{C/B}$ which is surjective as both modules are generated by differentials of elements in C.

The composition $B \to C \to \Omega_{C/A}$ is an A-derivation. This gives a map of B-modules $\Omega_{B/A} \to \Omega_{C/A}$ that induces the morphism of C-modules $\Omega_{B/A}\otimes_B C \to \Omega_{C/A}$. It remains to show exactness in the middle. For this consider the sub-C-module $N \subset \Omega_{C/A}$ generated by $\mathrm{d}(B)$. It is clear that $C \to \Omega_{C/A}/N$ is a B-derivations. This gives a morphism of C-modules $\Omega_{C/B} \to \Omega^0_{C/A}/N$ which is a section to the obvious morphism. Moreover, this morphism is surjective as both modules are generated by differentials of C. ☐

Proposition 4.34 — *Let A be a ring, B an A-algebra and $\mathfrak{b} \subset B$ an ideal. Then, there is an exact sequence of $B/\mathfrak{b}$-modules*

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B B/\mathfrak{b} \to \Omega_{(B/\mathfrak{b})/A} \to 0$$

where the first arrow sends $f + \mathfrak{b}^2$ to $\mathrm{d}_{B/A}(f) \otimes 1_{B/\mathfrak{b}}$.

Proof. Let $C = B/\mathfrak{b}$. We obviously have a surjective morphism $\Omega_{B/A} \to \Omega_{C/A}$ of B-modules which induces a surjective morphism of C-modules $\Omega_{B/A} \otimes_B C \to \Omega_{C/A}$.

Now we show that we have a well defined morphism

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B B/\mathfrak{b} \simeq \Omega_{B/A}/\mathfrak{b}\Omega_{B/A}.$$

Consider the composition

$$\alpha : \mathfrak{b} \overset{\mathrm{d}_{B/A}}{\to} \Omega_{B/A} \to \Omega_{B/A}/\mathfrak{b}\Omega_{B/A}$$

This an A-linear map. For $f, g \in \mathfrak{b}$, we have $\mathrm{d}_{B/A}(fg) = f\mathrm{d}_{B/A}g + g\mathrm{d}_{B/A}f \in \mathfrak{b}\Omega_{B/A}$. This shows that α send $\mathfrak{b}^2$ to 0. Thus, it factors trough $\mathfrak{b}/\mathfrak{b}^2$ yielding an A-linear map

$$\bar{\alpha} : \mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A}/\mathfrak{b}\Omega_{B/A}.$$

Next, we show that $\bar{\alpha}$ is a morphism of B-modules (and hence of $B/\mathfrak{b}$-modules). If $b \in B$ and $f \in \mathfrak{b}$, we have $\mathrm{d}_{B/A}(bf) = b\mathrm{d}_{B/A}f + f \in \mathrm{d}_{B/A}(b)$. As $f \in \mathrm{d}_{B/A}(b) \in \mathfrak{b}\Omega_{B/A}$, we get that $\bar{\alpha}(b.\bar{f}) = b\bar{\alpha}(\bar{f})$ where f is the class of f in $\mathfrak{b}/\mathfrak{b}^2$.

To finish the proof, it remains to show that the sequence of the statement is exact in the middle. First, note that the image of $\bar{\alpha}$ is generated by $\mathrm{d}_{B/A}f \otimes 1_C$ for $f \in \mathfrak{b}$.

As the image of f is zero in C, we see that the composition of the first two arrows in our sequence is zero. Next, consider

$$\Lambda = \frac{\Omega_{B/A}/\mathfrak{b}\Omega_{B/A}}{\mathrm{Im}(\bar{\alpha})}$$

endowed with $\mathrm{d}' : B \to \Lambda$. This again an A-derivation on B. Moreover, we have $\mathrm{d}'(\mathfrak{b}) = 0$. Thus, it induces $\mathrm{d}'' : B/\mathfrak{b} = C \to \Lambda$. By the universal property, we obtain $\Omega_{C/A} \to \Lambda$, which is a surjective section. This finishes the proof of the proposition. □

Proposition 4.35 — *Let A be a ring and B and A' two A-algebras. Let $B' = A' \otimes_A B$, considered as an A'-algebra. Then, $\Omega_{B/A} \otimes_A A' \simeq \Omega_{B'/A'}$.*

Proof. We have a derivation $B \to \Omega_{B'/A'}$ given by the composition $B \to B' \to \Omega_{B'/A}$. This yields a morphism of B-modules $\Omega_{B/A} \to \Omega_{B'/A'}$. As $\Omega_{B'/A'}$ is a B'-module, we get a morphism of B'-modules $\Omega_{B/A} \otimes_A A' \to \Omega_{B'/A'}$ which is surjective.

On the other hand, applying $A' \otimes_A -$ to $B \to \Omega_{B/A}$ yields an A'-derivation $B' \to \Omega_{B/A} \otimes_A A'$. This gives a morphism $\Omega_{B'/A'} \to \Omega_{B/A} \otimes_A A'$ which is also surjective and a section to the first morphism. □

Proposition 4.36 — *Let A be a ring and B an A-algebra. Let I be the kernel of the multiplication $B \otimes_A B \to B$. Then there is an isomorphism of B-modules $\Omega_{B/A} \simeq I/I^2$ sending $\mathrm{d}_{B/A}(f)$ to the class of $f \otimes 1 - 1 \otimes f$.*

Proof. As a B-module, I/I^2 is generated by the classes of $f \otimes 1 - 1 \otimes f$. Indeed, the ideal I is clearly generated by $a \otimes b - ab \otimes 1$. But, we may write this element as

$$1 \otimes b(a \otimes 1 - 1 \otimes a) - (ab \otimes 1 - 1 \otimes ab).$$

Moreover, $B \to I/I^2$ sending f to $f \otimes 1 - 1 \otimes f$ is a derivation as follows from:

$$fg \otimes 1 - 1 \otimes fg = (f \otimes 1)(g \otimes 1 - 1 \otimes g) + (1 \otimes g)(f \otimes 1 - 1 \otimes f).$$

Thus, by the universal property, there is a surjective B-linear map $\Omega_{B/A} \to I/I^2$.

Now, we construct an inverse to this map as follows. Consider $B \to \Omega_{B/A}$. This is A-linear and $\Omega_{B/A}$ is a B-module. Hence, it induces $B \otimes_A B \to \Omega_{B/A}$ sending $a \otimes b$ to $a\mathrm{d}_{B/A}(b)$. This map is B-linear for the action of B on the left at $B \otimes_A B$. We claim that the restriction of this to I^2 is zero. As I is generated as a B-module (for the action on the left) by $(a \otimes b - ab \otimes 1)$, it suffices to compute the images of $(a \otimes b - ab \otimes 1)(a' \otimes b' - a'b' \otimes 1)$. It is the image of $(aa' \otimes bb' - aa'b' \otimes b - aa'b \otimes b' + aa'bb' \otimes 1)$ Thus it is

$$aa'b\mathrm{d}b' + aa'b'\mathrm{d}b - aa'b'\mathrm{d}b - aa'b\mathrm{d}b' - 0 = 0.$$

We thus have a morphism $I/I^2 \to \Omega_{B/A}$. It is easy to check that this gives an inverse to the previously defined map. □

Corollary 4.37 — *Let A be a ring, B an A-module and $S \subset B$ a multiplicative subset. Then $S^{-1}\Omega_{B/A} \simeq \Omega_{S^{-1}B/A}$.*

4.7. **Sheaves of Kähler differentials on schemes.**

Let X be a scheme and $i : Y \to X$ a locally closed immersion. Let $U \subset X$ be an open subset containing the image of i and such that $i' : Y \to U$ is a closed immersion. We know that i' corresponds to a quasi-coherent sheaf of ideal $\mathcal{I} \subset \mathcal{O}_U$. Then, $\mathcal{N}_i = \mathcal{I}/\mathcal{I}^2$ is easily seen to be a quasi-coherent $\mathcal{O}_Y$-module. This is called the normal sheaf of the locally closed immersion i.

Definition 4.38 — *Let $f : Y \to X$ be a morphism of schemes. We define the sheaf of Kähler differentials of f to be the normal sheaf to the locally closed immersion $\Delta_f : Y \to Y \times_X Y$. This sheaf is denoted by Ω_f. We have an $f^*\mathcal{O}_X$-linear morphism of sheaves $\mathrm{d}_f : \mathcal{O}_Y \to \Omega_f$ on Y sending a function (defined on an open subset) to the class of $f \otimes 1 - 1 \otimes f$.*

From Corollary 4.37, we immediately deduce the following.

Lemma 4.39 — *If $Y = \mathsf{Spec}(B)$ and $X = \mathsf{Spec}(A)$, Ω_f is naturally isomorphic to the quasi-coherent sheaf associated to $\Omega_{B/A}$.*

Let A be a ring and M and A-module. We denote by $\mathrm{S}^n_A M$ the quotient of $M^{\otimes_A n} = M \otimes_A \cdots \otimes_A M$ by the tensors of the form:

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n - x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n.$$

Then $\bigoplus_{n=0}^{\infty} \mathrm{S}^n_A M$ is a commutative graded A-algebra which we denote by $A[M]$. When M is free of rank n, we obtain the A-algebra of polynomial in n variables.

This construction, can be generalized to schemes in the usual way. Let X be a scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. We define $\mathrm{S}^n_X \mathcal{M}$ as the sheaf associated to $U \rightsquigarrow \mathrm{S}^n_{\mathcal{O}_X(U)}(\mathcal{M}(U))$. This is again a quasi-coherent sheaf. Also $\mathcal{O}_X[\mathcal{M}] = \bigoplus_{n \in \mathbb{N}} \mathrm{S}^n_X \mathcal{M}$ is a graded quasi-coherent $\mathcal{O}_X$-algebra. We define two schemes:

- $\mathbb{V}(\mathcal{M}) = \mathsf{Spec}(\mathcal{O}_X[\mathcal{M}])$, the affine bundle associated to $\mathcal{M}$,
- $\mathbb{P}(\mathcal{M}) = \mathsf{Proj}(\mathcal{O}_X[\mathcal{M}])$, the projective bundle associated to $\mathcal{M}$.

Definition 4.40 — *With the notation above, we call a section of the affine bundle $\mathbb{V}(\mathcal{M})$ a morphism of schemes $X \to \mathbb{V}(\mathcal{M})$ such that the composition $X \to \mathbb{V}(\mathcal{M}) \to X$ is the identity.*

Lemma 4.41 — *To give a section of $\mathbb{V}(\mathcal{M})$ is equivalent to give a morphism of $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{O}_X$.*

Definition 4.42 — *Let $f : Y \to X$ be a morphism of schemes. The relative* tangent bundle *to f is the affine bundle associated to Ω_f. It is denoted by $\mathbf{T}_f$ and is endowed with a natural projection $\mathbf{T}_f \to Y$. As section of this projection is called a vector field. It corresponds to a derivation $\mathcal{O}_Y \to \mathcal{O}_Y$ which is $f^*\mathcal{O}_X$-linear.*

4.8. **Smooth schemes over a field.**

Definition 4.43 — *Let k be a field and X a k-scheme of finite type. Let $x \in |X|$ be a closed point. We say that X is* simple *(or* regular*) at x if the local ring $\mathcal{O}_{X,x}$ is regular. We say that X is* smooth *at x if for every finite extension $k \subset k'$ and any point $x' \in X' = X \otimes_k k'$ above x, X' is simple at x'.*

We say that X is smooth (resp. regular) if X is smooth (resp. regular) at all its closed points.

Lemma 4.44 — *Let X be a k-scheme of finite type and $x \in |X|$ a* rational point, *i.e., with $k \simeq \kappa(x)$. Then X is regular at x iff it is smooth at x.*

Proof. If X is smooth at x it is in particular regular at x. We need to prove the converse. Assume that X is regular at x.

Let k'/k be a finite extension and $X' = X \otimes_k k'$. The set of points $x' \in |X'|$ which are over $x \in |X|$, i.e., the inverse image of x along $X' \to X$, is identified with $\mathsf{Spec}(\kappa(x)) \times_X X' \simeq \mathsf{Spec}(\kappa(x) \otimes_k k')$. As $k \simeq \kappa(x)$, we see that $\kappa(x) \otimes_k k' \simeq k'$. This shows that there is unique point $x' \in |X'|$ which is mapped to x.

Thus, we need to show that $\mathcal{O}_{X',x'}$ is a regular local ring. We claim that $\mathcal{O}_{X,x} \otimes_k k' \simeq \mathcal{O}_{X',x'}$. As $\mathcal{O}_{X',x'}$ is the localization of $\mathcal{O}_{X,x} \otimes_k k'$ at a maximal ideal, it suffices to show that $\mathcal{O}_{X,x} \otimes_k k'$ is a local ring. But $\mathfrak{m}_x(\mathcal{O}_{X,x} \otimes_k k')$ is a maximal ideal as

$$\frac{\mathcal{O}_{X,x} \otimes_k k'}{\mathfrak{m}_x(\mathcal{O}_{X,x} \otimes_k k')} \simeq \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x} \otimes_k k' \simeq \kappa(x) \otimes_k k' \simeq k'$$

Moreover, as $\mathsf{Spec}(\mathcal{O}_{X,x} \otimes_k k') \to \mathsf{Spec}(\mathcal{O}_{X,x})$ is a finite morphism, every maximal ideal of $\mathcal{O}_{X,x} \otimes_k k'$ restricts to a maximal ideal of $\mathcal{O}_{X,x}$. This shows that $\mathfrak{m}_{x'} = \mathfrak{m}_x(\mathcal{O}_{X,x} \otimes_k k')$ is the unique maximal ideal of $\mathcal{O}_{X,x} \otimes_k k'$.

It remains to show that the local ring $\mathcal{O}_{X,x} \otimes_k k'$ is regular. Using again that $\mathsf{Spec}(\mathcal{O}_{X,x} \otimes_k k') \to \mathsf{Spec}(\mathcal{O}_{X,x})$ is a finite morphism, we see that $\mathrm{ht}(\mathfrak{m}_{x'}) = \mathrm{ht}(\mathfrak{m}_x)$. Call d this number. As $\mathcal{O}_{X,x}$ is regular, the ideal $\mathfrak{m}_x$ is generated by d elements $a_1, \dots, a_d$. It is then clear that $a_1 \otimes 1_{k'}, \dots, a_d \otimes 1_{k'}$ are generators of $\mathfrak{m}_{X',x'}$. This finishes the proof of the lemma. □

Proposition 4.45 — *Let $X = \mathsf{Spec}(A)$ be an affine scheme of finite type over a field k. Let $x \in |X|$ be a closed point and denote $d = \dim_x(X)$. We fix a presentation (i.e., a surjective morphism) $p : k[t_1, \dots, t_n] \twoheadrightarrow A$ and a generating family $(f_1, \dots, f_m)$ of $\ker(p)$. The following conditions are equivalent:*

(a) *X is smooth at x,*
(b) *The dimension of the $\kappa(x)$-vector space $\Omega_{X/k} \otimes_{\mathcal{O}_X} \kappa(x)$ is equal to d,*
(b′) *The dimension of the $\kappa(x)$-vector space $\Omega_{X/k} \otimes_{\mathcal{O}_X} \kappa(x)$ is less than d,*
(c) *The rank of the Jacobian matrix $J(x) = \left(\frac{\partial f_i}{\partial t_j}(x)\right)_{(i,j) \in [\![1,m]\!] \times [\![1,n]\!]}$ is equal to $n - d$,*
(c′) *The rank of the Jacobian matrix $J(x) = \left(\frac{\partial f_i}{\partial t_j}(x)\right)_{(i,j) \in [\![1,m]\!] \times [\![1,n]\!]}$ is bigger than $n - d$.*

Proof. We first show that $(b) \Leftrightarrow (c)$ and $(b') \Leftrightarrow (c')$. Let $I = (f_1, \dots, f_m) = \ker(p)$. We have an exact sequence of A-modules

$$I/I^2 \to \bigoplus_{j=1}^{n} A\mathrm{d}t_j \to \Omega_{A/k} \to 0.$$

The first morphism sends f_i to $\mathrm{d}f_i = \sum_{j=1}^n \frac{\partial f_i}{\partial t_j} \mathrm{d}t_i$. Thus, we have an exact sequence

$$\bigoplus_{i=1}^{m} A \xrightarrow{J} \bigoplus_{j=1}^{n} A \to \Omega_{A/k} \to 0$$

Where J is the Jacobian matrix. Applying $- \otimes_A \kappa(x)$, we get the exact sequence

$$\bigoplus_{i=1}^{m} \kappa(x) \xrightarrow{J(x)} \bigoplus_{j=1}^{n} \kappa(x) \to \Omega_{A/k} \otimes_A \kappa(x) \to 0.$$

This proves our claim.

As $(b) \Rightarrow (b')$ and $(c) \Rightarrow (c')$, it remains to show that $(a) \Rightarrow (c)$ and $(c') \Rightarrow (a)$.

So assume that X is smooth at x. To check (c), we may extend the scalar to any finite extension k'/k. Thus, we may assume that x is a rational point of X, i.e., $k = \kappa(x)$. As $\mathrm{ht}(\mathfrak{m}_x) = d$, $\mathcal{O}_{X,x}$ is regular if and only if $\dim_{\kappa(x)} \mathfrak{p}_x/\mathfrak{p}_x^2 = d$, with $\mathfrak{p}_x \subset A$ the maximal ideal corresponding to x. Let $\mathfrak{q}_x \subset k[t_1, \ldots, t_n]$ be the inverse image of $\mathfrak{p}_x$. We have an exact sequence

$$I/I^2 \otimes_A \kappa(x) \xrightarrow{P} \mathfrak{q}_x/\mathfrak{q}_x^2 \to \mathfrak{p}_x/\mathfrak{p}_x^2 \to 0.$$

As $\mathfrak{q}_x$ is a maximal ideal and $k \simeq \kappa(x)$, we have $\mathfrak{q}_x = (t_1 - a_1, \ldots, t_n - a_n)$ for $a_i \in k$. Thus, $\mathfrak{q}_x/\mathfrak{q}_x^2$ is of dimension n. It follows that P is of rank $n - d$. Now, we have a commutative diagram

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^m A & \xrightarrow{\ \ J\ \ } & & & & & \\
\downarrow & & \searrow & & & & \\
I/I^2 & \longrightarrow & \bigoplus_{j=1}^n A\mathrm{d}t_j & \longrightarrow & \Omega_{A/k} & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{q}_x/\mathfrak{q}_x^2 & \xrightarrow{u} & \bigoplus_{j=1}^n \kappa(x)\mathrm{d}t_j & \longrightarrow & \Omega_{\kappa(x)/k} & &
\end{array}$$

As x is a rational point, $\Omega_{\kappa(x)/k} = 0$. Thus u is surjective and hence an isomorphism by rank considerations. It follows that the rank of P is equal to the rank of $J(x)$.

To end the proof, we still need to show that $(c') \Rightarrow (a)$. As (c') is invariant by base change through a finite extension k'/k, we only need to show that X is regular at x. The above diagram, show that the rank of P is at least $n - d$. Hence $\dim_{\kappa(x)} \mathfrak{p}_x/\mathfrak{p}_x^2 \leq d$. As X is of dimension d at x, this shows that $\mathcal{O}_{X,x}$ is regular. $\square$

Corollary 4.46 — *Let X be a finite type k-scheme. Let $x \in X$ be a smooth point of X. Then, there is an open neighborhood of x in X which is smooth over k.*

Proof. We may assume that $X = \mathsf{Spec}(A)$ is affine. Let $d = \dim_x(X)$. As $\mathcal{O}_{X,x}$ is regular, it is in particular an integral domain. Hence, replacing X by an affine open neighborhood of x, we may assume that A is integral. In this case, for every $y \in X$, we have $d = \dim_y(X) = \mathrm{degtr}(\mathrm{Frac}(A)/k)$.

Next, as $\dim_{\kappa(x)} \Omega_{A/k} \otimes_A \kappa(x) = d$, by Nakayama's Lemma, $\Omega_{A/k} \otimes_A \mathcal{O}_{X,x}$ is generated by less than d elements. Hence, replacing X by a small affine neighborhood of x, we may assume that $\Omega_{A/k}$ is generated by less than d elements. But then, for every $y \in X$, $\dim_{\kappa(y)}(\Omega_{A/k} \otimes_A \kappa(y)) \leq d$. Thus, condition (b') of Proposition 4.45 is satisfied. This show that X is smooth over k. $\square$

Corollary 4.47 — *Let X be a finite type k-scheme every where of dimension d. Then the following conditions are equivalent:*

(i) *X is smooth,*

(ii) $\Omega_{X/k}$ *is locally free of rank* d.

Proof. Obviously, (i) implies (b) of Proposition 4.45, and hence (ii). Next, we show that (ii)⇒(i). As $\Omega_{X/k}$ is an $\mathcal{O}_X$-module of finite type, the implication (ii)⇒(i) follows from (b) of Proposition 4.45 and the Lemma below. □

Lemma 4.48 — *Let* X *be a reduced finite type* k*-scheme and* $\mathcal{M}$ *a quasi-coherent* $\mathcal{O}_X$*-module of finite type. Assume that for every closed point* $x \in |X|$*, the* $\kappa(x)$*-vector space* $\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ *has dimension* d*. Then* $\mathcal{M}$ *is locally free of rank* d*.*

Proof. Fix $x \in |X|$ a closed point. It suffices to show that there is an open neighborhood of x over which $\mathcal{M}$ is free of rank d. We may assume that $X = \mathsf{Spec}(A)$ is affine, x correspond to a maximal ideal $\mathfrak{p} \subset A$ and $\mathcal{M} = \widetilde{M}$ for an A-module M.

As $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})$ is a vector space which is generated by d elements, we may find by Nakayama's lemma a surjection $A_\mathfrak{p}^d \twoheadrightarrow M_\mathfrak{p}$. Inverting the denominators in A, we may assume that we have a surjection

$$p : A^d \twoheadrightarrow M.$$

Let $N = \ker(p)$. We need to show that $N = 0$. Assume the contrary. Let $(a_1, \ldots, a_d) \in A^d$ be a non-zero element of N. Assume that $a_1 \neq 0$. As A is reduced, a_1 is not nilpotent and A_a is not empty. Hence, there exist prime ideals of A which do not contains a. If $\mathfrak{q}$ is maximal such prime ideal, it is also a maximal ideal of A (use that $\dim_\mathfrak{q}(\mathsf{Spec}(A)) = \dim_\mathfrak{q}(\mathrm{D}(a)))$.

Now, applying $- \otimes_A A/\mathfrak{q}$, we get an exact sequence

$$N/\mathfrak{q}N \to (A/\mathfrak{q})^d \to M/\mathfrak{q}M \to 0$$

As the image of the first map contains $(a_1 + N, \ldots, a_d + N)$, it is not zero. This implies that $M/\mathfrak{q}M$ has dimension less that $d-1$. This is a contradiction. □

Proposition 4.49 — *Let* $s : Y \to X$ *be a closed immersion of smooth* k*-schemes. Then, we have a short exact sequence of locally free* $\mathcal{O}_Y$*-modules*

$$0 \to \mathcal{N}_s \to s^\star\Omega_{X/k} \to \Omega_{Y/k} \to 0.$$

Proof. We may assume that $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(A/I)$. Let $B = A/I$. Then, we have an exact sequence

$$I/I^2 \to \Omega_{A/k}/I\Omega_{A/k} \to \Omega_{B/k} \to 0.$$

We need to show exactness on the left. We may assume that the kernel of $\Omega_{A/k}/I \to \Omega_{B/k}$ is free. Let $f_1, \ldots, f_c \in I$ such that the images of $f_i + I^2$ form a basis. It suffices to show that $I_1 = I$ around $s(Y)$, with $I_1 = (f_1, \ldots, f_c)$.

Let $B_1 = A/I_1$. We have an exact sequence

$$I_1/I_1^2 \to \Omega_{A/k} \otimes_A B_1 \to \Omega_{B_1/k} \to 0.$$

It follows that $\Omega_{B_1/k} \otimes_{B_1} B \simeq \Omega_{B/k}$. As $\dim(B) \leq \dim(B_1)$, we see that B_1 is smooth at every point of B. This implies that $\mathsf{Spec}(B)$ is a disjoint union of connected components of $\mathsf{Spec}(B_1)$. □

Recall that a field k is perfect if for any irreducible polynomial $P \in k[t]$ we have $P' \neq 0$. Thus, every field k of characteristic zero is perfect. A field k of characteristic $p > 0$ is perfect if and only if the Frobenius morphism $(-)^p : k \to k$ is invertible. Indeed, let P be a polynomial of $k[t]$. Then, the condition $P' = 0$ is equivalent to the fact that $P = Q(t^p)$ for some $Q = \sum_{i=0}^n a_i t^i$. If the Frobenius morphism is invertible, we may find b_i such that $b_i^p = a_i$. But then, $P = (\sum_{i=0}^n b_i t^i)^p$ which contradicts that P is irreducible.

Lemma 4.50 — *Let k be a field and $P \in k[t]$ an irreducible polynomial. Let $k' = k[t]/P$. Then $\Omega_{k'/k} = 0$ if and only if $P' \neq 0$. In particular, over a perfect field, the spectrum of a finite extension is smooth.*

Proof. Indeed, as $\dim(\mathsf{Spec}(k')) = 0$, the condition $\Omega_{k'/k} = 0$ holds if and only if the rank of the 1×1-matrix $P' + (P) \in k'$ is 1. Thus, $P' \neq 0$ is a necessary condition. It is also sufficient. Indeed, as $P' \neq 0$ and P is irreducible, we see that P and P' generate the ideal $k[t]$. Hence, $P' + (P)$ is invertible in k'. ☐

Proposition 4.51 — *Let k be a perfect field. Then any reduced finite type k-scheme X admits a dense open subscheme which is smooth over k.*

Proof. We may assume that $X = \mathsf{Spec}(A)$ is affine and irreducible, hence integral. Let $K = \mathrm{Frac}(A)$ and $d = \mathrm{degtr}(K/k)$. By the Lemma below, we may find an inclusion of k-extensions $k(t_1, \ldots, t_d) \subset K$ which is a separable extension, i.e., can be generated by an element with minimal polynomial having a non-zero derivative.

We have an exact sequence

$$\Omega_{k(t_1,\ldots,t_d)/k} \otimes_{k(t_1,\ldots,t_d)} K \to \Omega_{K/k} \to \Omega_{K/k(t_1,\ldots,t_d)} \to 0$$

As $K/k(t_1, \ldots, t_d)$ is a separable extension, $\Omega_{K/k(t_1,\ldots,t_d)} = 0$ and hence the first arrow is surjective. But, the $k(t_1, \ldots, t_n)$-vector space $\Omega_{k(t_1,\ldots,t_n)}$ has $(\mathrm{d}t_i)_{i=1,\ldots,d}$ as a basis and hence has dimension d. It follows that $\Omega_{K/k}$ is a K-vector space of dimension less than d. Now, $\Omega_{K/k} = \Omega_{A/k} \otimes_A K$. As $\Omega_{A/k}$ is a finitely generated A-module, we may find a non-empty affine open subset U of $\mathsf{Spec}(A)$ over which $\Omega_{X/k}$ is free of rank less than d. Thus condition (b') of Proposition 4.45 is satisfied for any point of U. This shows that U is smooth over k. ☐

Lemma 4.52 — *Let k be a perfect field, and K/k a finitely generated extension. Then, there exists an injection $k(t_1, \ldots, t_d) \subset K$ which is a separable finite extension.*

Proof. Let p be the characteristic of k. When $p = 0$, there is nothing to show. Thus, we may assume $p > 0$.

We argue by contradiction. Assume the contrary and choose $k(t_1, \ldots, t_d) \subset K$ such that $[K : K^s]$ is minimal with K^s the sub-extension of separable elements over $k(t_1, \ldots, t_d)$, i.e., the element having a minimal polynomial with non-zero derivative. Let $K^i \subset K$ be the sub-extension consisting of the totally inseparable elements, i.e., those element having a minimal polynomial with zero derivative (check that K^s and K^i are indeed subfields of K!). We have $K = K^s K^i$ and $[K^s : k(t_1, \ldots, t_d)] = [K : K^s]$.

Let $u \in K^i \backslash k(t_1, \ldots, t_d)$. As k is perfect, the maximal inseparable extension of $k(t_1, \ldots, t_d)$ is given by $\bigcup_{r \geq 1} k(t_1^{1/p^r}, \ldots, t_d^{1/p^r})$. Thus, we may find q, a power of p, such that $u^q \in k(t_1, \ldots, t_d)$. We assume that q is minimal and write $Q = u^q$.

We may find $i \in [\![1, d]\!]$ such that $\frac{\partial Q}{\partial t_i} \neq 0$. Otherwise, as k is perfect, Q has a p-root R and $u^{q/p} = R$ contradicting the minimality of q. It follows that the extension $k(t_1, \ldots, \hat{t_i}, \ldots, t_d, u)[t_i]/(Q - u^p)$ is a separable extension of $k(t_1, \ldots, \hat{t_i}, \ldots, t_d, u)$. We get a contradiction with the minimality of $[K : K^s]$. □

4.9. **Smooth and étale morphisms.**

Definition 4.53 — *Let $f : Y \to X$ be a morphism of schemes. We say that f is* smooth *if f is flat, of finite presentation and if for every $x \in X$, the scheme theoretic fiber $Y_x = Y \times_X \mathsf{Spec}(\kappa(x))$ is a smooth $\kappa(x)$-scheme.*

We say that f is étale *if f is smooth and quasi-finite.*

Lemma 4.54 — *Smooth and étale morphisms are stable by base change.*

Proof. This is obvious. □

Proposition 4.55 — *Let f be a smooth morphism. Then Ω_f is locally free. Suppose we are given a commutative triangle*

$$\begin{array}{ccc} Y & \xrightarrow{s} & X \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\ & & T \end{array}$$

with s a closed immersion and f and g smooth. Then, we have an exact sequence of locally free $\mathcal{O}_Y$-modules

$$0 \to \mathcal{N}_s \to s^\star\Omega_f \to \Omega_g \to 0 \tag{12}$$

where $\mathcal{N}_s$ is the normal sheaf to the immersion s.

Proof. We may assume that f comes from a morphism of algebras $A \to B$. We choose a presentation $A[t_1, \ldots, t_n] \twoheadrightarrow B$ with kernel I. We have an exact sequence of B-modules

$$I/I^2 \to \bigoplus_{i=1}^{n} B\mathrm{d}t_i \to \Omega_{B/A} \to 0. \tag{13}$$

We have a short exact sequence of A-modules

$$0 \to I \to A[t_1, \ldots, t_n] \to B \to 0.$$

As B is a flat A-algebra, we deduce that I is a flat A-module. It follows that for $\mathfrak{p} \in \mathsf{Spec}(A)$, the morphism $I \otimes_A \kappa(\mathfrak{p}) \to I\kappa(\mathfrak{p})[t_1, \ldots, t_n]$ is invertible. Moreover, the image of $I^2 \otimes_A \kappa(\mathfrak{p}) \to I \otimes_A \kappa(\mathfrak{p})$ is identified via this isomorphism with $I^2\kappa(\mathfrak{p})[t_1, \ldots, t_n]$. It follows that we have an isomorphism

$$(I/I^2) \otimes_A \kappa(\mathfrak{p}) \simeq \frac{I\kappa(\mathfrak{p})[t_1, \ldots, t_n]}{I^2\kappa(\mathfrak{p})[t_1, \ldots, t_n]}.$$

Moreover, we know that $\Omega_{B/A} \otimes_A \kappa(\mathfrak{p}) \simeq \Omega_{B\otimes_A\kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$. It follows that when we apply $- \otimes_A \kappa(\mathfrak{p})$ to (13), we get up to a natural isomorphism

$$0 \to \frac{I\kappa(\mathfrak{p})[t_1, \ldots, t_n]}{I^2\kappa(\mathfrak{p})[t_1, \ldots, t_n]} \to \bigoplus_{i=1}^{n} \kappa(\mathfrak{p})\mathrm{d}t_i \to \Omega_{B\otimes_A\kappa(\mathfrak{p})/\kappa(\mathfrak{p})} \to 0$$

which is a short exact sequence by Proposition 4.49.

Now, let $N = \ker((I/I^2) \to \oplus_{i=1}^n B\mathrm{d}t_i))$ and $K = (I/I^2)/N$. As $I/I^2 \otimes_B \kappa(y) \to K \otimes_B \kappa(y)$ is surjective, we see that

$$0 \to K \to \bigoplus_{i=1}^n B\mathrm{d}t_i \to \Omega_{B/A} \to 0$$

ia also an exact sequence after applying $- \otimes_B \kappa(\mathfrak{q})$ for any $\mathfrak{q} \in \mathsf{Spec}(B)$. This implies that $\mathrm{tor}^1_B(\Omega_{B/A}, \kappa(\mathfrak{q})) = 0$ and that $\Omega_{B/A}$ is flat, and hence locally free (as it is finitely presented).

For the second statement, we use the argument before to show that the sequence (12) becomes a short exact sequence after applying $- \otimes_A \kappa(\mathfrak{p})$ for every $\mathfrak{p} \in \mathsf{Spec}(A)$. This implies that (12) is a short exact sequence. □

Theorem 4.56 — *Let $f : Y \to X$ be morphism of finite presentation. The following conditions are equivalent:*

(i) *f is étale,*
(ii) *f is flat and $\Omega_f = 0$,*
(iii) *f is flat and $\Delta_f : Y \to Y \times_X Y$ is an open immersion,*
(iv) *For every $y \in Y$, $V = \mathsf{Spec}(B)$ an affine neighborhood of y and $U = \mathsf{Spec}(A)$ an affine neighborhood of $f(V)$, the A-algebra B is isomorphic to $A[t_1, \dots, t_n]/(f_1, \dots, f_n)$ with the Jacobian matrix $\mathrm{J}(f_1, \dots, f_n)$ invertible on every point of V.*

Proof. The implication (ii)⇒(i) is easy. Indeed, as the scheme theoretic fibers $f^{-1}(x)$ are empty or smooth of dimension zero, we have $(\Omega_f)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y) \simeq (\Omega_{f^{-1}(x)/\kappa(x)})_y \otimes \mathcal{O}_{f^{-1}(x),y}\kappa(y) = 0$ for every $y \in f^{-1}(x)$. As Ω_f is locally free, we have $\Omega_f = 0$.

Let's show that (ii)⇔(iii). If $\Delta_f : Y \to Y \times_X Y$ is an open immersion, then its normal sheaf is zero. This gives that $\Omega_f = 0$. Conversely, assume that $\Omega_f = 0$. To show, that Δ_f is an open immersion, we may assume that $X = \mathsf{Spec}(A)$, $Y = \mathsf{Spec}(B)$. Let I be the kernel of the multiplication map $B \otimes_A B \to B$. We know that $I = I^2$. It follows that for every $\mathfrak{q} \in \mathsf{Spec}(B \otimes_A B)$, $I_{\mathfrak{q}}$ is either 0 or $(B \otimes_A B)_{\mathfrak{q}}$...

Next we show that (ii)⇒(iv). Choose a presentation $A[t_1, \dots, t_n] \to B$ with kernel I. We have an exact sequence

$$0 \to I/I^2 \xrightarrow{\theta} \bigoplus_{i=1}^n B\mathrm{d}t_i \to \Omega_{B/A} \to 0.$$

As $\Omega_{B/A} = 0$, we see that θ is invertible. Let $f_i \in I$ be elements such that $\theta(f_i + I^2) = \mathrm{d}t_i$. It is sufficient to show that $I = (f_1, \dots, f_n)$. Indeed, the Jacobian matrix $\mathrm{J}(f_1, \dots, f_n)$ is the equal to the identity on every point of $\mathsf{Spec}(B)$. Let $I_1 = (f_1, \dots, f_n)$. The morphism $I_1/I_1^2 \to I/I^2$ is surjective and hence $I = I_1 + I^2$. This implies that $I_g = (I_1)_g$ for some $g \in A[t_1, \dots, t_n]$ such that $\mathsf{Spec}(B) \subset \mathrm{D}(g)$. Thus, we may take the following presentation $A[t_1, \dots, t_n, s]/(f_1, \dots, f_n, gs-1) \simeq B$ which clearly satisfies to (iv).

To end the proof, we still need to show that (iv)⇒(i). The main point is to show that B is a flat A-algebra. For this, we may assume that A is noetherian, local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{n} \subset B$ be a maximal ideal above $\mathfrak{m}$. It is sufficient to show that $\hat{B} = \mathrm{Lim}_n B/\mathfrak{n}^n$ is a flat $\hat{A} = \mathrm{Lim}_n A/\mathfrak{m}^n$-algebra. But we have more precisely that

$\hat{A} \simeq \hat{B}$. Indeed, we may assume that $f_i(0,\ldots,0) = 0$. Then $\mathrm{Jac}(f_1,\ldots,f_n)(0,\ldots,0)$ invertible, implies that $t_i \rightsquigarrow f_i$ is an automorphism of the ring $\hat{A}[\![t_1,\ldots,t_n]\!]$. □

Corollary 4.57 — *Let $f : Y \to X$ be a morphism of schemes of finite presentation. The following two conditions are equivalent.*

(1) *f is smooth.*
(2) *For every $y \in Y$, there exists affine open neighborhoods V and U of $y \in Y$ and $x = f(y) \in X$ with $f(V) \subset U$, and a factorization*

$$V \xrightarrow{e} \mathbb{A}^d_U \xrightarrow{p} U, \qquad f = p \circ e$$

with e étale and p the obvious projection.

Proof. Clearly (2)⇒(1). We show the converse implication. We may assume that $X = \mathsf{Spec}(A)$ and $Y = \mathsf{Spec}(B)$. Fix a presentation $A[t_1,\ldots,t_n] \to B$ with kernel I. We then have an exact sequence of B-modules

$$0 \to I/I^2 \to \bigoplus_{i=1}^{n} B\mathrm{d}t_i \to \Omega_{B/A} \to 0.$$

We may assume that I/I^2 is free and that it admits a complementary sub-module in $\bigoplus_{i=1}^{n} B\mathrm{d}t_i \to \Omega_{B/A}$ freely generated by a subset of the basis $\{\mathrm{d}t_1,\ldots,\mathrm{d}t_n\}$, say $\{\mathrm{d}t_1,\ldots,\mathrm{d}t_r\}$.

Next, let $f_{r+1},\ldots,f_n \in I$ be elements such that $f_{r+1}+I^2,\ldots,f_n+I^2$ form a basis of I/I^2. Then we may find $g \in A[t_1,\ldots,t_n]$ such that $\mathsf{Spec}(B) \subset \mathrm{D}(g)$ and $I_g = (f_{r+1},\ldots,f_n)_g$. Hence, replacing our initial presentation by $A[t_1,\ldots,t_n,s] \to B$ sending s to the inverse of the image of g, we may assume that $I_1 = I$. But then $\mathsf{Spec}(B) \to \mathsf{Spec}(A[t_1,\ldots,t_r])$ is étale as it satisfies to the condition (iv) of Theorem 4.56. This proves the corollary. □

Proposition 4.58 — *Assume we are given a commutative triangle of schemes*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow^{h} & \downarrow{\scriptstyle g} \\ & & T. \end{array}$$

If h and g are étale, then f is also étale.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow{\scriptstyle \mathrm{id}\times f} & & \downarrow{\scriptstyle \Delta_g} \\ Y \times_T X & \longrightarrow & X \times_T X \\ & \searrow^{pr_2} & \downarrow{\scriptstyle pr_2} \\ & & X. \end{array}$$

As g is étale, Δ_g is an open immersion. Hence $\mathrm{id} \times f : Y \to Y \times_T X$ is also an open immersion. Moreover, $pr_2 : Y \times_T X \to X$ is a base change of h, hence it is étale. As f is equal to the composition $pr_2 \circ (\mathrm{id} \times f)$, we are done. $\square$

5. Cohomology of Quasi-Coherent Sheaves

5.1. Some notions of abstract category theory.

Definition 5.1 — *A* category $\mathcal{C}$ *consists of the following data:*

(c1) *a class* $\mathrm{ob}(\mathcal{C})$*, whose elements are called the* objects *of* $\mathcal{C}$*,*

(c2) *for* A, $B \in \mathrm{ob}(\mathcal{C})$*, a set* $\hom_{\mathcal{C}}(A,B)$*, whose elements are called the* morphisms *or* arrows *from* A *to* B*,*

(c3) *for* A, B, $C \in \mathrm{ob}(\mathcal{C})$ *an application*

$$\hom_{\mathcal{C}}(A,B) \times \hom_{\mathcal{C}}(B,C) \to \hom_{\mathcal{C}}(A,C) \tag{14}$$

called the composition*.*

A morphism $a \in \hom_{\mathcal{C}}(A,B)$*, will be denoted* $a : A \to B$*. Given two morphisms* $a : A \to B$ *and* $b : B \to C$*, we denote* $b \circ a : A \to C$ *the image of* (a,b) *by* (14)*. With these notation, the following two properties should hold:*

(i) *For every* $A \in \mathrm{ob}(\mathcal{C})$*, there exists an element* $\mathrm{id}_A : A \to A$ *which is a unit for the composition, i.e., for every* $b : B \to A$ *and* $c : A \to C$*, we have* $\mathrm{id}_A \circ b = b$ *and* $c \circ \mathrm{id}_A = c$*.*

(ii) *The composition is associative, i.e., for every morphisms* $a : A \to B$*,* $b : B \to C$ *and* $c : C \to D$*, we have* $c \circ (b \circ a) = (c \circ b) \circ a$*.*

Example 5.2 — There are plenty of examples of categories:

(1) $\mathcal{S}et$: where objects are sets and morphisms are applications.

(2) $\mathcal{T}op$: where objects are topological spaces and morphisms are continuous maps.

(3) $\mathcal{R}ing$: where objects are commutative rings and morphisms are morphisms of rings.

(4) $\mathcal{S}ch$: where objects are schemes and morphisms given by morphisms of locally ringed spaces.

(5) $\mathcal{G}rp$: where objects are groups.

(6) $\mathcal{A}b$: where objects are abelian groups.

(7) If A is a ring, $\mathcal{M}od(A)$: where objects are A-modules.

(8) If S is a scheme, $\mathcal{M}od(S)$: where objects are quasi-coherent $\mathcal{O}_S$-modules.

(9) ...

An arrow $a : A \to B$ in a category $\mathcal{C}$ is called *invertible* (or an *isomorphism*) if there exists $b : B \to A$ such that $a \circ b = \mathrm{id}_B$ and $b \circ a = \mathrm{id}_A$. Such a b, if it exists, is unique and is called the inverse of a and denoted a^{-1}. We usually denote $a : A \xrightarrow{\sim} B$ an arrow which is invertible.

Definition 5.3 — *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. A* functor F *from* $\mathcal{C}$ *to* $\mathcal{D}$*, usually denoted by* $F : \mathcal{C} \to \mathcal{D}$*, consists of the following data:*

(f1) *an application* $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$*,*

(f2) *for* A, $B \in \mathrm{ob}(\mathcal{C})$ *an application* $F : \hom_{\mathcal{C}}(A,B) \to \hom_{\mathcal{D}}(F(A), F(B))$*.*

Moreover, the following two conditions should be satisfied:

(i) *for all* $A \in \mathrm{ob}(\mathcal{C})$*,* $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$*,*

(ii) *for all* $a : A \to B$ *and* $b : B \to C$ *in* $\mathcal{C}$*,* $F(b \circ a) = F(b) \circ F(a)$*.*

Given a category $\mathcal{C}$, we can define a new category $\mathcal{C}^\circ$ having the same objects as $\mathcal{C}$ but where the arrows are reversed, i.e., $\hom_{\mathcal{C}^\circ}(A, B) = \hom_{\mathcal{C}}(B, A)$. We clearly have $\mathcal{C}^{\circ\circ} = \mathcal{C}$. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is by definition a functor $F : \mathcal{C}^\circ \to \mathcal{D}$.

Example 5.4 — There are also plenty of examples of functors:

(1) $\mathcal{S}et \to \mathcal{T}op$ which takes a set S to the S itself endowed with the discrete topology.
(2) $\mathcal{T}op \to \mathcal{S}et$ which takes a topological space S to its underlying set.
(3) $\mathcal{G}rp \to \mathcal{S}et$ which takes a group to its underlying set.
(4) $\mathcal{T}op \to \mathcal{R}ing$ which takes a topological space X to the ring $\mathcal{C}^0(X)$ of continuous real functions on X.
(5) $\mathcal{R}ing \to \mathcal{S}ch$ which takes a ring A to the affine scheme $\mathsf{Spec}(A)$.
(6) ...

Definition 5.5 — *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories and* $F, G : \mathcal{C} \to \mathcal{D}$*. A* natural transformation $t : F \to G$ *is a collection of morphisms* $t_A : F(A) \to G(A)$ *for every* $A \in \mathrm{ob}(\mathcal{C})$ *such that the squares*

$$\begin{array}{ccc} F(A) & \xrightarrow{t_A} & G(A) \\ {\scriptstyle F(a)}\downarrow & & \downarrow{\scriptstyle G(a)} \\ F(B) & \xrightarrow{t_B} & G(B) \end{array}$$

commute for every arrow $a : A \to B$ *in* $\mathcal{C}$*. The set of natural transformation is denoted by* $\hom(F, G)$*.*

The following result is known as the *Yoneda Lemma*:

Lemma 5.6 — *Let* $\mathcal{C}$ *be a category. For* $X \in \mathrm{ob}(\mathcal{C})$*, we define a functor* $\mathrm{h}_X : \mathcal{C}^\circ \to \mathcal{S}et$ *by sending* $A \in \mathrm{ob}(\mathcal{C})$ *to the set* $\mathrm{h}_X(A) = \hom_{\mathcal{C}}(A, X)$*. Moreover, given a functor* $h : \mathcal{C}^\circ \to \mathcal{E}ns$*, there is a natural bijection*

$$h(X) \simeq \hom(\mathrm{h}_X, \mathrm{h}).$$

between the set $h(X)$ *to and the set of natural transformations from* h_X *to* h*.*

Proof. Given $a : A \to B$, we have an application $(-) \circ a : \hom(B, X) \to \hom(A, X)$, which we may write $\mathrm{h}_X(a) : \mathrm{h}_X(B) \to \mathrm{h}_X(A)$. If a is an identity, so is $\mathrm{h}_X(a)$. To see that $\mathrm{h}_X(-)$ preserves the composition of arrows, let $b : B \to C$ be another arrow. The formula $\mathrm{h}_X(a \circ b) = \mathrm{h}_X(b) \circ \mathrm{h}_X(a)$, is equivalent to $(-) \circ (a \circ b) = ((-) \circ a) \circ b$ which is true by the associativity of the composition.

Now, let $s \in h(X)$. For $A \in \mathrm{ob}(C)$, we have an application $\hat{s}_A : \hom_{\mathcal{C}}(A, X) = \mathrm{h}_X(A) \to h(A)$, which takes an arrow $u : A \to X$ to the element $h(u)(s) \in h(A)$. Let's check that the family $(\hat{s}_A)_{A \in \mathrm{ob}(\mathcal{C})}$ defines a natural transformation $\hat{s} : \mathrm{h}_X \to h$. Take $a : A \to B$. We need to check that the square

$$\begin{array}{ccc} \mathrm{h}_X(B) & \xrightarrow{\hat{s}_B} & h(B) \\ {\scriptstyle \mathrm{h}_X(a)}\downarrow & & \downarrow{\scriptstyle h(a)} \\ \mathrm{h}_X(A) & \xrightarrow{\hat{s}_A} & h(A) \end{array}$$

commutes. But, for $v : B \to X$, we have $\hat{s}_A \circ \mathrm{h}_X(a)(v) = \hat{s}_A(v \circ a) = h(v \circ a)(s) = h(a) \circ h(v)(s) = h(a) \circ \hat{s}_B(v)$.

The assignment $s \rightsquigarrow \hat{s}$ gives an application $\alpha : h(X) \to \hom(\mathrm{h}_X, h)$. It remains to show that this application is bijective. We define a map

$$\beta : \hom(\mathrm{h}_X, h) \to h(X)$$

in the opposite direction by sending a natural transformation $t : \mathrm{h}_X \to h$ to $t_X(\mathrm{id}_X) \in h(X)$. It is clear that $\beta \circ \alpha = \mathrm{id}$. We next show that $\alpha \circ \beta = \mathrm{id}$. In other words, every natural transformation $t : \mathrm{h}_X \to h$ is of the form $\hat{s}$ where $s = t_X(\mathrm{id}_X)$. Let $u : A \to X$ be an arrow. As t is a natural transformation, we should have a commutative square

$$\begin{array}{ccc} \mathrm{h}_X(X) & \xrightarrow{t_X} & h(X) \\ {\scriptstyle \mathrm{h}_X(u)}\downarrow & & \downarrow{\scriptstyle h(u)} \\ \mathrm{h}_X(A) & \xrightarrow{t_A} & h(A). \end{array}$$

It follows that $t_A(u) = t_A \circ \mathrm{h}_X(u)(\mathrm{id}_X) = h(u) t_X(\mathrm{id}_X) = h(u)(s) = \hat{s}_A(u)$. This ends the proof of the lemma. ☐

Definition 5.7 — *Let $\mathcal{C}$ be a category and $h : \mathcal{C}^\circ \to \mathcal{E}ns$ a functor. We say that h is* representable *if there exists an object $X \in \mathcal{C}$ and an invertible natural transformation $\mathrm{h}_X \simeq h$.*

Remark 5.8 — We can be more precise about the isomorphism in Definition 5.7. Indeed, using Yoneda's Lemma, we see that h is representable iff there exists a couple (X, s) with $X \in \mathrm{ob}(\mathcal{C})$ and $s \in h(X)$ such that the natural transformation $\hat{s} : \mathrm{h}_X \to h$ is invertible.

Lemma 5.9 — *With the notation of* Definition 5.7*, the object X, if it exists, is unique up to a unique isomorphism.*

Proof. Let X and X' be two such couples. We may form the composition

$$\mathrm{h}_X \xrightarrow{\sim} h \xrightarrow{\sim} \mathrm{h}_{X'}.$$

This is an invertible natural transformation and, by Yoneda's Lemma, it is induced by an invertible arrow $u \in \mathrm{h}_{X'}(X) = \hom_{\mathcal{C}}(X, X')$. This proves the lemma. ☐

Example 5.10 — Let A be a commutative ring. Consider the functor $h : \mathcal{S}ch^\circ \to \mathcal{E}ns$ which takes a scheme X to the set $\hom_{\mathcal{R}ing}(A, \mathcal{O}_X(|X|))$. Then h is representable by $\mathsf{Spec}(A)$. Indeed, by Proposition 2.25, to give a morphism of rings from A to $\mathcal{O}_X(|X|)$ is equivalent to give a morphism of schemes from X to $\mathsf{Spec}(A)$.

If X is an object of $\mathcal{C}$, we define h°_X to be the functor $\hom_{\mathcal{C}}(X, -)$. It is the representable functor associated to X in the opposite category $\mathcal{C}^\circ$. So everything said about h_X has a dual statement for h°_X. In particular, we have the notion of a corepresentable functor.

Example 5.11 — Let A be a ring and B an A-algebra. The functor $\mathrm{Der}_A(B, -) : \mathcal{M}od(B) \to \mathcal{S}et$ which takes a B-module M to the set of derivations from B to M is corepresentable by the B-module $\Omega_{B/A}$.

5.2. **Projective limits in an abstract category.**

We first begin with of projective limits in the category of sets. Let $(I, \leq)$ be a partially ordered set. A *projective system* of sets indexed by I is a collection of sets S_i, one for every $i \in I$, and applications $s_{ij} : S_i \to S_j$ whenever $i \leq j$. We also impose that the triangle

$$\begin{array}{ccc} S_i & \xrightarrow{s_{ij}} & S_j \\ & {\scriptstyle s_{ik}}\searrow & \downarrow{\scriptstyle s_{jk}} \\ & & S_k \end{array}$$

commutes whenever $i \leq j \leq k$.

Here is another equivalent way to say what is a projective system of sets. We may view the ordered set $(I, \leq)$ as a category by setting $\mathrm{ob}(I) = I$ and

$$\hom_I(i, j) = \begin{cases} \emptyset & \text{if } i \not\leq j, \\ \{*\} & \text{if } i \leq j. \end{cases}$$

Then a projective system is simply a functor from I to $\mathcal{S}et$.

A projective system will be often denoted by $(S_i)_{i \in I}$ when the maps s_{ij} are understood.

Definition 5.12 — *We denote $\varprojlim_{i\in I} S_i$ the subset of $\prod_{i\in I} S_i$ whose elements are the families $(a_i)_{i\in I}$ with $a_i \in S_i$ and such that $a_j = s_{ij}(a_i)$ whenever $i \leq j$. The set $\varprojlim_{i\in I} S_i$ is called the projective limit of the projective system $(S_i)_{i\in I}$.*

Example 5.13 — Assume that the set I is endowed with the discrete ordering, i.e., $i \leq j$ if and only if $i = j$. Then a projective system $(S_i)_{i\in I}$ is simply a family of sets indexed by I. Moreover, $\varprojlim_{i\in I} S_i = \prod_{i\in I} S_i$.

If I is the empty set and S is the empty family, $\varprojlim_{i\in I} S_i$ is the set having exactly one element.

Example 5.14 — Let $I = \{a, b, c\}$ with the ordering $a \leq c$ and $b \leq c$. To give a projective system indexed by I, is equivalent to give three sets S_a, S_b and S_c together with two applications $S_a \to S_c$ and $S_b \to S_c$. Moreover, the projective limit of such a system is simply the fiber product $S_a \times_{S_c} S_b$.

Lemma 5.15 — *Let $(I, \leq)$ be a partially ordered set, and $S = (S_i)_{i\in I}$ and $T = (T_i)_{i\in I}$ two projective systems. Let $S \to T$ be a morphism of projective system, i.e., a natural transformation between the functors $S : I \to \mathcal{E}ns$ and $T : I \to \mathcal{E}ns$. Then there is a natural application $\varprojlim_{i\in I} S_i \to \varprojlim_{i\in I} T_i$.*

Proof. This is an easy exercise. □

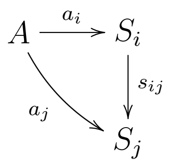
Now, let $\mathcal{C}$ be a category. We define the notion of a projective family in $\mathcal{C}$ as in the case of sets. It is a collection of objects $S_i \in \mathrm{ob}(\mathcal{C})$ and morphisms $s_{ij} : S_i \to S_j$ whenever $i \leq j$. Equivalently, it is a functor $S : I \to \mathcal{C}$. Let define a contravariant functor

$$\text{“}\varprojlim_{i\in I}\text{”} S_i : \mathcal{C} \to \mathcal{E}ns$$

by sending an object A of $\mathcal{C}$ to the set $\varprojlim_{i\in I} \hom_{\mathcal{C}}(A, S_i)$. Given $a : A \to B$, the morphisms $\hom_{\mathcal{C}}(B, S_i) \to \hom_{\mathcal{C}}(A, S_i)$ yields a morphism of projective systems of sets and hence an application $\varprojlim_{i\in I} \hom_{\mathcal{C}}(B, S_i) \to \varprojlim_{i\in I} \hom_{\mathcal{C}}(A, S_i)$.

Definition 5.16 — *Let $(I, \leq)$ be a partially ordered set and $(S_i)_{i \in I}$ a projective system in a category $\mathcal{C}$. We say that the projective limit of $(S_i)_{i \in I}$ exists if the contravariant functor "$\varprojlim$"$_{i \in I} S_i$ is representable by an object of $\mathcal{C}$ which we denote $\varprojlim_{i \in I} S_i$ and which we call the* projective limit *of the system $(S_i)_{i \in I}$.*

Lemma 5.17 — *Let $(I, \leq)$ be a partially ordered set and $(S_i)_{i \in I}$ be a projective system in a category $\mathcal{C}$. Assume that the projective limit of $(S_i)_{i \in I}$ exists. Then the object $\varprojlim_{i \in I} S_i$ of $\mathcal{C}$ satisfies to the following universal property. For every object $A \in \mathcal{C}$, to give a morphism $A \to \varprojlim_{i \in I} S_i$ is equivalent to give a family of morphisms $a_i : A \to S_i$ such that for every $i \leq j$, the triangle*



commutes.

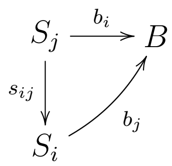
Proof. This is obvious from the definition. □

5.3. **Injective limits in an abstract category.**

The notion of injective systems is dual to the notion of projective systems. Let $(I, \leq)$ be a partially ordered set. An injective system in a category $\mathcal{C}$ is a collection of objects S_i and morphisms $S_j \to S_i$ whenever $i \leq j$. Equivalently, an injective system in $\mathcal{C}$ is a projective system in the opposite category $\mathcal{C}^\circ$.

Definition 5.18 — *Let $(I, \leq)$ be a partially ordered set and $(S_i)_{i \in I}$ an injective system in a category $\mathcal{C}$. We say that $(S_i)_{i \in I}$ has an injective limit if the projective system $(S_i)_{i \in I}$ of $\mathcal{C}^\circ$ has a projective limit which we denote by $\varinjlim_{i \in I} S_i$.*

Lemma 5.19 — *Let $(I, \leq)$ be a partially ordered set and $(S_i)_{i \in I}$ be an injective system in a category $\mathcal{C}$. Assume that the injective limit of $(S_i)_{i \in I}$ exists. Then the object $\varinjlim_{i \in I} S_i$ satisfies to the following universal property. For every object $B \in \mathcal{C}$, to give a morphism $\varinjlim_{i \in I} S_i \to B$ is equivalent to give a family of morphisms $b_i : S_i \to B$ such that for every $i \leq j$, the triangle*



commutes.

Proof. This is obvious from the definition. □

Proposition 5.20 — *Injective limits exist in the category $\mathcal{S}et$.*

Proof. Let $(I, \leq)$ be a partially ordered set and $(S_i)_{i \in I}$ be an injective system of sets. Let $E = \coprod_{i \in I} S_i$ be the disjoint union of the S_i. An element of E will be denoted as (i, s) where $i \in I$ and $s \in S_i$. Let $\sim$ be the smallest equivalence relation on E such that $(i, s) \simeq (j, t)$ if $i \leq j$ and $s_{ij}(t) = s$. Then $E/\sim$ is easily seen to satisfies to the universal property of Lemma 5.19. □

Definition 5.21 — *Let $(I, \leq)$ be a partially ordered set. We say that $(I, \leq)$ is cofiltered if I is not empty and if for every i, $j \in I$, there exists $k \in I$ such that $k \leq i$ and $k \leq j$.*

Remark 5.22 — Let $(I, \leq)$ be cofiltered partially ordered set and $(S_i)_{i \in I}$ an inductive system of sets. Let $E = \coprod_{i \in I} S_i$ endowed with the equivalence relation of the proof of Proposition 5.20. Then $(i_1, s_1) \sim (i_2, s_2)$ iff there exists $i_3 \leq i_1$, i_2 such that $s_{i_3 i_1}(s_1) = s_{i_3 i_2}(s_2)$. If follows from that that when S_i are (abelian) groups and s_{ij} morphism of (abelian) groups, the set $\varinjlim_{i \in I} S_i$ is naturally an (abelian) group.

Proposition 5.23 — *In $\mathcal{S}et$, cofiltered injective limits commute with finite projective limits. More precisely, let $(I, \leq)$ and $(J, \leq)$ be two partially ordered sets, which we view as categories. Let $S : I^\circ \times J \to \mathcal{S}et$ be a functor and denote $S_{i,j}$ the set $S(i,j)$ for $(i,j) \in I \times J$. Then there is a natural application*

$$\varinjlim_{i \in I} \varprojlim_{j \in J} S_{i,j} \to \varprojlim_{j \in J} \varinjlim_{i \in I} S_{ij}. \tag{15}$$

Moreover, if I is cofiltered and J is finite, this application is bijective.

Proof. Fix $i_0 \in I$. Then, for every $j \in J$, we have applications $S_{i_0,j} \to \varinjlim_{i \in I} S_{i,j}$. They form a morphism of projective systems. Hence an application

$$\varprojlim_{j \in J} S_{i_0,j} \to \varprojlim_{j \in J} \varinjlim_{i \in I} S_{i,j}.$$

We obtain the morphism of the statement using Lemma 5.19.

Now, assume that I is cofiltered and J finite. We first show that (15) is injective. Let's take two elements of $\varinjlim_{i \in I} \varprojlim_{j \in J} S_{i,j}$ having the same image by (15). These two elements are equivalence classes of $[(i, (x_j)_{j \in J})]$ and $[(i', (x'_j)_{j \in J})]$ with $(x_j)_{j \in} J$ and $(x'_j)_{j \in J}$ in $\varprojlim_j S_{i,j}$ and $\varprojlim_j S_{i',j}$. As I is cofiltered, we may assume that $i = i'$. To say that our two elements have the same image by (15) is equivalent to say that for every $j \in J$, there exists $i_j \leq i$ such that $s_{i_j,i}(x_i) = s_{i_j,i}(x'_i)$. As I is cofiltered and J is finite, we may find $k \in I$ which is smaller than all the i_j. But then our two elements can be written as $[(k, (s_{ki}(x_j))_j]$ and $[(k, (s_{ki}(x'_j))_j)]$. Thus they are equal.

We now show that (15) is surjective. An element of $\varprojlim_j \varinjlim_i S_{ij}$ is a family $([(i_j, x_j)])_j$. As I is filtered and J is finite, we may write our family as $([i, x_j])_{j \in J}$. We should have $s_{jk}([i, x_j]) = [i, x_k]$ whenever $j \leq k$. Thus, there should exists $i_{j \leq k} \leq i$ such that $s_{i_{j \leq k},i}(x_j) = s_{i_{j \leq k},i}(x_k)$. As J is finite, we can find i' smaller than all the $i_{j \leq k}$. But then, our element can be written as $[(i', s_{i'i}(x_j))]_j$ and is clearly in the image of (15). ☐

5.4. Application to sheaves of sets.

Let X be a topological space. We may view a presheaf of sets on X as a functor $\mathrm{Ouv}(X)^\circ \to \mathcal{S}et$ where Ouv is the ordered set of opens in X. Let $PSh(X)$ denotes the category of presheaves of sets on X and $Sh(X) \subset PSh(X)$ the subcategory of sheaves. Given an open subset $U \subset X$, we denote $\Gamma(U, -) : PSh(X), Sh(X) \to \mathcal{S}et$ the functor which takes a presheaf F to the set $F(U)$.

Lemma 5.24 — *Projective and injective limits exists in the category $PSh(X)$ and they commute with the functors $\Gamma(U, -)$.*

Proof. Let $(I,\leq)$ be a partially ordered set and $(F_i)_{i\in I}$ a projective system of presheaves. Let G be the presheaf which associate to U the set $\varprojlim_{i\in I} F_i(U)$. We need to check that G is the projective limit of our system. We will check the universal property. Note that we have obvious morphisms $G\to F_i$.

Let E be a presheaf of sets on X. Let $E\to G$ be a morphism. We deduce morphisms $E\to F_i$ by composing with $G\to F_i$. Moreover, $(E\to F_i)_{i\in I}$ is obviously an element of $\varprojlim_{i\in I}\hom(E,F_i)$.

Conversely, let $(E\to F_i)_{i\in I}$ be an element of $\varprojlim_{i\in I}\hom(E,F_i)$. Then for any $U\subset X$ open, $(E(U)\to F_i(U))_{i\in I}$ is an element of $\varprojlim_{i\in I}\hom(E(U),F_i(U))$. Hence a canonical morphism $E(U)\to\varprojlim_{i\in I}F_i(U)=G(U)$. This gives a morphism $E\to G$.

The case of injective limits is treated in the same way. □

Lemma 5.25 — *Let $(I,\leq)$ be a partially ordered set and $(F_i)_{i\in I}$ a projective system of presheaves of sets. Assume that for all i, F_i is a sheaf. Then $\varprojlim_{i\in I}F_i$ is also a sheaf which is the projective limit of $(F_i)_{i\in I}$ in the category $Sh(X)$.*

Proof. It is clear that $\varprojlim_{i\in I}F_i$ is separated. Indeed, let $U\subset X$ be an open subset and $(U_\alpha)_{\alpha\in A}$ an open covering of X. Let $(x_i)_{\in I}$ and $(y_i)_{i\in I}$ be two sections of $(\varprojlim_{i\in I}F_i)(U)=\varprojlim_{i\in I}F_i(U)$. Now assume that $(x_i)_{\in I}$ and $(y_i)_{i\in I}$ have the same restriction on the U_α, i.e., $((x_i)_{|U_\alpha})_i=((y_i)_{|U_\alpha}$. It follows that for all $i\in I$, $(x_i)_{|U_\alpha}=(y_i)_{|U_\alpha}$. As F_i is separated, we get that $x_i=y_i$ for all i.

Now, we check the gluing property for $\varprojlim_{i\in I}F_i$. Let U and U_α be as before. Assume that we have a locally defined section $((x_{i,\alpha})_{i\in I})_{\alpha\in A}$. Fixing i and using that F_i is a sheaf, we get a section $x_i\in F_i(U)$ such that $(x_i)_{|U_\alpha}=x_{i,\alpha}$. Moreover, for $i\leq j$, $s_{ij}(x_i)_{|U_\alpha}=s_{ij}(x_{i,\alpha})=x_{j,\alpha}=s_{ij}(x_j)_{|U_\alpha}$. Using that F_j is separated, we deduce that $(x_i)_{i\in I}$ is an element of $\varprojlim_{i\in I}F_i(U)$. □

The previous result is false in general for injective limits, i.e., an injective limit of sheaves (computed in $PSh(X)$) is not a sheaf in general. However, we have the following:

Lemma 5.26 — *Injective limits exists in the category $Sh(X)$. More precisely, given an injective system $(F_i)_{i\in I}$ of sheaves of sets on X, the sheaf $\mathrm{a}(\varinjlim_{i\in I}F_i)$ associated to the presheaf $\varinjlim_{i\in I}F_i$ is the injective limit of $(F_i)_{i\in I}$.*

Proof. Let G be a sheaf. To give a morphism in $Sh(X)$ from $\mathrm{a}(\varinjlim_{i\in I}F_i)$ to G, is equivalent to give a morphism in $PSh(X)$ from $\varinjlim_{i\in I}F_i$ to G, which in turn, is equivalent to a family of morphism (in $Sh(X)$) $(F_i\to G)_{i\in I}$ such that $F_j\to G$ is the composition $F_j\to F_i\to G$ for all $i\leq j$. This proves the lemma. □

Recall that a topological space X is called quasi-compact if every open for covering $(U_\alpha)_{\alpha\in A}$, there exists a finite subset $A_0\subset A$ such that $X=\bigcup_{\alpha\in A_0}U_\alpha$.

Proposition 5.27 — *Let $(I,\leq)$ be a cofiltered partially ordered set. Let $(F_i)_{i\in I}$ be an inductive system of sheaves on a topological space X. Assume that every open subset of X is quasi-compact. Then $\varinjlim_{i\in I}F_i$ is a sheaf, and so is the injective limit of $(F_i)_{i\in I}$ is the category $Sh(X)$.*

Proof. Let G a presheaf. Let $U \subset X$ be an open subset, and $(U_\alpha)_{\alpha\in A}$ an open covering of U. Let $(A_2, \leq)$ be the partially ordered set whose elements are subsets $\{\alpha,\beta\} \subset A$ (where possibly $\alpha = \beta$) and where $\{\alpha,\beta\} \leq \{\alpha',\beta'\}$ if $\{\alpha,\beta\} \subset \{\alpha',\beta'\}$. We have a projective system $(G(U_\alpha \cap U_\beta))_{\{\alpha,\beta\}\in A_2}$ whose projective limit is exactly the set of locally defined sections on U with respect to the cover $(U_\alpha)_\alpha$. It follows that G is a sheaf if and only if

$$G(U) \to \varprojlim_{\{\alpha,\beta\}\in A_2} G(U_\alpha \cap U_\beta)$$

is a bijection.

We apply the previous discussion for $G = \varinjlim_{ißI} F_i$. We need to show that for any $U \subset X$ and any cover $(U_\alpha)_{\alpha\in I}$, which we may assume to be finite, the canonical morphism

$$\varinjlim_{i\in I} F_i(U) \to \varprojlim_{\{\alpha,\beta\}\in A_2} \varinjlim_{i\in I} F_i(U_\alpha \cap U_\beta).$$

As I is cofiltered and A_2 is finite, we may use Proposition 5.23 to rewrite this morphism as

$$\varinjlim_{i\in I} F_i(U) \to \varinjlim_{i\in I} \varprojlim_{\{\alpha,\beta\}\in A_2} F_i(U_\alpha \cap U_\beta).$$

Our claim now follows as

$$F_i(U) \to \varprojlim_{\{\alpha,\beta\}\in A_2} F_i(U_\alpha \cap U_\beta)$$

is invertible for all i, F_i being a sheaf. □

5.5. **Abelian categories.**

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is pointed if it has an initial object $\emptyset$, a final object $\star$ and if the unique morphism $\emptyset \to \star$ is invertible. We the denote 0 an initial or a final object of $\mathcal{C}$. Given two objects A and B in $\mathcal{C}$, we define the zero morphism $0 \in \hom_{\mathcal{C}}(A,B)$ as being the composition $A \to 0 \to B$.

Moreover, given two objects A and B in $\mathcal{C}$, there is an obvious morphism

$$A \coprod B \to A \prod B \tag{16}$$

which is equivalently given by the couple $(A \coprod B \to A, A \coprod B \to B)$ or $(A \to A \prod B, B \to A \prod B)$ where the morphisms are the compositions

$$A \coprod B \to A \coprod 0 \simeq A \quad \text{and} \quad A \coprod B \to 0 \coprod B \simeq B,$$

$$A \simeq A \prod 0 \to A \prod B \quad \text{and} \quad B \simeq 0 \prod B \to A \prod B.$$

Definition 5.28 — *A* pre-additive category *is a category $\mathcal{C}$ which is pointed and where* (16) *is invertible for all A and B in $\mathcal{C}$.*

A functor $F : \mathcal{C} \to \mathcal{D}$ between pre-additive categories is called pre-additive (or simply additive) if F commutes with finite products and coproducts.

Proposition 5.29 — *Let $\mathcal{C}$ be a pre-additive category. Then for A and B in $\mathcal{C}$, $\hom_{\mathcal{C}}(A,B)$ is naturally a commutative monoide. Moreover, the composition in $\mathcal{C}$ is distributive.*

Moreover, if $F : \mathcal{C} \to \mathcal{D}$ is an additive functor, then F induces a morphism of monoids on morphisms.

Proof. Let $f,\ g : A \to B$ be two arrows in $\mathcal{C}$. We define $f+g$ as being the composition one of the two compositions:

$$A \xrightarrow{\Delta} A \prod A \xrightarrow{f \prod g} B \prod B \simeq B \coprod B \xrightarrow{\nabla} B$$

$$A \xrightarrow{\Delta} A \prod A \simeq A \coprod A \xrightarrow{f \coprod g} B \coprod B \xrightarrow{\nabla} B.$$

We leave it as an exercise to check that the law $(f, g) \to f + g$ is associative, commutative and has a zero element given by the zero morphism. That the composition in $\mathcal{C}$ is distributive for this law, is easy. □

Definition 5.30 — *An* additive category *is a pre-additive category* $\mathcal{C}$ *such that for every* A *and* B *in* $\mathcal{C}$*, the monoid* $\hom_{\mathcal{C}}(A, B)$ *is a group, i.e., every element* $f \in \hom_{\mathcal{C}}(A, B)$ *has an inverse* $(-f)$ *for the addition.*

Example 5.31 — The category $\mathcal{A}b$ of abelian groups is an additive category. Indeed, if A and B are abelian groups, $A \coprod B = A \oplus B$ and $A \prod B = A \times B$, etc.

Actually, $\mathcal{A}b$ is an abelian category (see Definition 5.33 below).

Let $\mathcal{C}$ be an additive category. Let $f : A \to B$ be a morphism in $\mathcal{C}$. The kernel of f, if it exists, is the projective limit of the following diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ A & \xrightarrow{f} & B, \end{array}$$

(i.e., the fiber product $A \times_B 0$) and will be denoted by $\ker(f)$. There is an obvious morphism $\ker(f) \to A$ which is characterized by the following universal property. For every $a : N \to A$ such that $f \circ a = 0$, there exists a unique morphism $a_0 : N \to \ker(A)$ such that a is the composition $(\ker(A) \to A) \circ a_0$.

Dually, the cokernel of f, it it exists, is the injective limit of the following diagram

$$\begin{array}{ccc} A & \longrightarrow & 0, \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

(i.e., the amalgamated sum $B \coprod_A 0$) and will be denoted by $\operatorname{coker}(f)$. There is an obvious morphism $B \to \operatorname{coker}(f)$ which is characterized by the following universal property. For every $b : B \to K$ such that $b \circ f = 0$, there exists a unique morphism $b_0 : \operatorname{coker}(f) \to B$ such that b is the composition $b_0 \circ (B \to \operatorname{coker}(f))$.

Definition 5.32 — *A morphism* $f : A \to B$ *is called a monomorphism (resp. epimorphism) if* $\ker(f) = 0$ *(resp.* $\operatorname{coker}(f) = 0$*). We also say that* f *is injective (resp. surjective). Equivalently,* f *is a monomorphism if for every* $a : N \to A$*, the condition* $f \circ a = 0$ *implies that* $a = 0$*. Similarly,* f *is an epimorphism if for every* $b : B \to K$*, the condition* $b \circ f = 0$ *implies that* $b = 0$*.*

Monomorphisms, (resp. epimorphisms) will be denote as $f : A \hookrightarrow B$ *(resp.* $f : A \twoheadrightarrow B$*).*

It is easy to see that for a general morphism $f : A \to B$, $\ker(f) \to A$ is a monomorphism and $B \to \operatorname{coker}(f)$ is an epimorphism.

We now define the image of $f : A \to B$ to be the kernel of $B \to \mathrm{coker}(f)$ and the coimage to be the cokernel of $\ker(f) \to A$:

$$\mathrm{im}(f) = \ker(B \to \mathrm{coker}(f)) \qquad \text{and} \qquad \mathrm{coim}(f) = \mathrm{coker}(\ker(f) \to A).$$

By construction, there is canonical morphisms $\mathrm{im}(f) \to B$ and $A \to \mathrm{coim}(f)$. Moreover, as the compositions $\ker(f) \to A \to B$ and $A \to B \to \mathrm{coker}(f)$ are zeros, we have by the universal properties of the kernels and cokernels, canonical factorizations of f:

$$\begin{array}{ccc} A & \twoheadrightarrow & \mathrm{coim}(f) \\ & {\scriptstyle f} \searrow & \downarrow \\ & & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \longrightarrow & \mathrm{im}(f) \\ & {\scriptstyle f} \searrow & \downarrow \\ & & B. \end{array}$$

Moreover, as $A \to B \to \mathrm{coker}(f)$ is zero and $A \to \mathrm{coim}(f)$ is an epimorphism, we deduce that $\mathrm{coim}(f) \to B \to \mathrm{coker}(f)$ is also zero. Hence, we deduce a morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$. Moreover, f is the composition

$$A \twoheadrightarrow \mathrm{coim}(f) \to \mathrm{im}(f) \hookrightarrow B.$$

Definition 5.33 — *An abelian category is an additive category $\mathcal{A}$ such that for every morphism $f : A \to B$ has a kernel and a cokernel and such that the canonical morphism* $\mathrm{coim}(f) \to \mathrm{im}(f)$ *is invertible.*

Lemma 5.34 — *Let $\mathcal{A}$ be an additive category where every morphism has a kernel and a cokernel. Then $\mathcal{A}$ is abelian if and only if every morphism $f : A \to B$ can be factored by an epimorphism followed with a monomorphism*

$$A \twoheadrightarrow \bullet \hookrightarrow B.$$

Proof. This is an easy exercise. □

5.6. Complexes in abelian categories.

Definition 5.35 — *Let $\mathcal{C}$ be an additive category. A* complex *in $\mathcal{C}$ is a collection $(A^i, d^i)_{i\in\mathbb{Z}}$ where A^i are objects in $\mathcal{A}$ and $d^i : A^i \to A^{i+1}$ are morphisms such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. The morphisms d^i are called the differentials.*

We denote such a complex $(A^\bullet, d^\bullet)$ or sometimes $A^\bullet$ if no confusion can arise. We say that $A^\bullet$ is bounded below (resp. above) if $A^i = 0$ for i small (resp. big) enough. We say that $A^\bullet$ is bounded if $A^i = 0$ except for finitely many values of i.

A morphism of complexes $f^\bullet : (A^\bullet, d^\bullet) \to (A'^\bullet, d'^\bullet)$ is a collection $f^i : A^i \to B^i$ such that the squares

$$\begin{array}{ccc} A^i & \xrightarrow{d^i} & A^{i+1} \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle f^{i+1}} \\ A'^i & \xrightarrow{d'^i} & A'^{i+1} \end{array}$$

commute for all $i \in \mathbb{Z}$.

Let $\mathcal{A}$ be an abelian category and $(A^\bullet, d^\bullet)$ a complex. As $d^i \circ d^{i-1} = 0$, there is a canonical morphism $\mathrm{im}(d_{i-1}) \hookrightarrow \ker(d_i)$ making the square

$$\begin{array}{ccc} A^{i-1} & \xrightarrow{d^{i-1}} & A^i \\ \downarrow & & \uparrow \\ \mathrm{im}(d_{i-1}) & \hookrightarrow & \ker(d_i) \end{array}$$

commutative.

Definition 5.36 — *For $i \in \mathbb{Z}$, we define*

$$\mathrm{H}^i(A^\bullet) = \mathrm{coker}(\mathrm{im}(d_{i-1}) \to \ker(d_i))) = \frac{\ker(d_i)}{\mathrm{im}(d_{i-1})}.$$

This is the i-th cohomology *object of the complex $A^\bullet$.*

Lemma 5.37 — *Let $f : A^\bullet \to A'^\bullet$ be a morphism of complexes in an abelian category. Then there is a canonical morphism*

$$\mathrm{H}^i(f) : \mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(A'^\bullet).$$

Proof. We have a commutative diagram

$$\begin{array}{ccccc} A^{i-1} & \xrightarrow{d^{i-1}} & A^i & \xrightarrow{d^i} & A^{i+1} \\ \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} \\ A'^{i-1} & \xrightarrow{d'^{i-1}} & A'^i & \xrightarrow{d'^i} & A'^{i+1}. \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{im}(d^{i-1}) & \hookrightarrow & A^i & \xrightarrow{d^i} & A^{i+1} \\ \downarrow & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} \\ \mathrm{im}(d'^{i-1}) & \hookrightarrow & A'^i & \xrightarrow{d'^i} & A'^{i+1}. \end{array}$$

and thus, a commutative square

$$\begin{array}{ccc} \mathrm{im}(d^{i-1}) & \hookrightarrow & \ker(d^i) \\ \downarrow & & \downarrow \\ \mathrm{im}(d'^{i-1}) & \hookrightarrow & \ker(d'^i) \end{array}$$

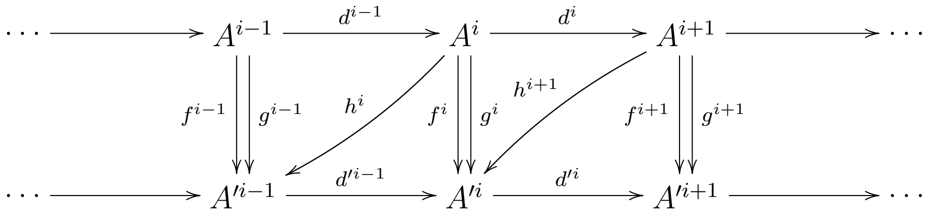
which finally gives $\mathrm{H}^i(f)$. □

Definition 5.38 — *A morphism of complexes $f : A^\bullet \to A'^\bullet$ is called a quasi-isomorphism if it induces isomorphisms on cohomology, i.e., for every $i \in \mathbb{Z}$, $\mathrm{H}^i(f) : \mathrm{H}^i(A) \to \mathrm{H}^i(A')$ is invertible.*

Definition 5.39 — *Let f, g : $A^\bullet \to A'^\bullet$ be a morphism of complexes. A* homotopy *from f to g is a collection of morphisms $h^i : A^i \to A'^{i-1}$ such that*

$$f^i - g^i = h^{i+1} \circ d^i + d'^{i-1} \circ h^i.$$

We picture a homotopy as follows:



We say that f and g are homotopic *if there exists a homotopy form f to g.*

Lemma 5.40 — *Let $A^\bullet$ and $A'^\bullet$ be two complexes. Then the homotopy relation on* $\hom(A^\bullet, A'^\bullet)$ *is an equivalence relation which is compatible with composition of morphisms (on the left and on the right).*

Proof. This is an easy exercise. □

Proposition 5.41 — *Let f, $g : A^\bullet \to A'^\bullet$ be two morphisms of complexes in $\mathcal{A}$. If f and g are homotopic, then they induce the same morphism on cohomology, i.e.,* $\mathrm{H}^i(f) = \mathrm{H}^i(g)$ *for all $i \in \mathbb{Z}$.*

Proof. We call $\tilde{f}^i$, $\tilde{g}^i : \mathrm{im}(d^{i-1}) \to \mathrm{im}(d'^{i-1})$ and $\hat{f}^i$, $\hat{g}^i : \ker(d^i) \to \ker(d'^i)$, the morphisms induced from f and g.

We obviously have a commutative diagram

$$\begin{array}{ccccc} \mathrm{im}(d^{i-1}) & \hookrightarrow & \ker(d^i) & \hookrightarrow & A^i \\ \downarrow{\scriptstyle \tilde{f}^i-\tilde{g}^i} & & \downarrow{\scriptstyle \hat{f}^i-\hat{g}^i} & & \downarrow{\scriptstyle f^i-g^i} \\ \mathrm{im}(d'^{i-1}) & \hookrightarrow & \ker(d'^i) & \hookrightarrow & A'^i \end{array}$$

Now recall that $f^i - g^i = h^{i+1} \circ d^i + d'^{i-1} \circ h^i$. As d^i vanishes on $\ker(d^i)$, it follows that

$$\begin{array}{ccccc} \mathrm{im}(d^{i-1}) & \hookrightarrow & \ker(d^i) & \hookrightarrow & A^i \\ \downarrow{\scriptstyle \tilde{f}^i-\tilde{g}^i} & & \downarrow{\scriptstyle \hat{f}^i-\hat{g}^i} & & \downarrow{\scriptstyle d'^{i-1}h^i} \\ \mathrm{im}(d'^{i-1}) & \hookrightarrow & \ker(d'^i) & \hookrightarrow & A'^i \end{array}$$

It follows that $\mathrm{im}(\hat{f}^i - \hat{g}^i) \subset \mathrm{im}(d'^{i-1})$. This implies that the composition

$$\ker(d^i) \to \ker(d'^i) \to \frac{\ker(d'^i)}{\mathrm{im}(d'^{i-1})}$$

is zero, and thus also

$$\frac{\ker(d^i)}{\mathrm{im}(d^{i-1})} \to \frac{\ker(d'^i)}{\mathrm{im}(d'^{i-1})}.$$

This proves the proposition. □

Corollary 5.42 — *Let $(A^\bullet, d^\bullet)$ be a complex in an abelian category $\mathcal{A}$. Assume that the identity of $A^\bullet$ is homotopic to identity. Then* $\mathrm{H}^i(A^\bullet) = 0$ *for all $i \in \mathbb{Z}$.*

Proof. Indeed, the identity map on $\mathrm{H}^i(A)$ is zero. □

Definition 5.43 — *Let $f : A^\bullet \to A'^\bullet$. We say that f is a homotopy equivalence, if there exists $g : A'^\bullet \to A^\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity.*

Corollary 5.44 — *A homotopy equivalence is a quasi-isomorphism.*

5.7. The cone of a morphism of complexes.

Let $f^\bullet : A^\bullet \to A'^\bullet$ be a morphism of complexes. We define a new complex $(\mathrm{C}^\bullet(f), d_f^\bullet)$ as follows. For $i \in \mathbb{Z}$, we set $\mathrm{C}^i(f) = A'^i \oplus A^{i+1}$ and

$$d_f^i = \begin{pmatrix} d'^i & f^{i+1} \\ 0 & -d^{i+1} \end{pmatrix} : A'^i \oplus A^{i+1} \to A'^{i+1} \oplus A^{i+2}.$$

The simple computation

$$\begin{pmatrix} d'^{i+1} & f^{i+2} \\ 0 & -d^{i+2} \end{pmatrix} \circ \begin{pmatrix} d'^i & f^{i+1} \\ 0 & -d^{i+1} \end{pmatrix} = \begin{pmatrix} d'^{i+1} \circ d'^i & d'^{i+1} \circ f^{i+1} - f^{i+2} \circ d^{i+1} \\ 0 & d^{i+2} \circ d^{i+1} \end{pmatrix} = 0$$

shows that $(\mathrm{C}^\bullet(f), d_f^\bullet)$ is indeed a complex.

Definition 5.45 — *The complex* $(\mathrm{C}^\bullet(f), d_f^\bullet)$ *defined above is called the* mapping cone *or simply the* cone *of f.*

Given a complex $(A^\bullet, d^\bullet)$, we define a new complex $(A^\bullet[+1], d^\bullet[+1])$ by setting $A^i[+1] = A^{i+1}$ and $d^i[+1] = -d^{i+1}$. With the above notations, we have canonical morphisms of complexes $A'^\bullet \to \mathrm{C}^\bullet(f)$ and $\mathrm{C}^\bullet(f) \to A^\bullet[+1]$. The sequence

$$A^\bullet \xrightarrow{f} A'^\bullet \xrightarrow{\alpha_f} \mathrm{C}^\bullet(f) \xrightarrow{\beta_f} A^\bullet[+1]$$

is called a distinguished triangle.

Lemma 5.46 — *The composition $\beta_f \circ \alpha_f$ is zero. The composition $\alpha_f \circ f$ is homotopic to zero.*

Proof. The first claim is obvious. For the second one, we remark that $\alpha_f \circ f$ is given in degree i by

$$\begin{pmatrix} f^i \\ 0 \end{pmatrix} : A^i \to A'^i \oplus A^{i+1}.$$

Consider the homotopy (h^i) given by

$$h^i = \begin{pmatrix} 0 \\ \mathrm{id}_{A^i} \end{pmatrix} : A^i \to A'^{i-1} \oplus A^i.$$

The following computation

$$\begin{pmatrix} d'^{i-1} & f^i \\ 0 & -d^i \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_{A^i} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathrm{id}_{A^{i+1}} \end{pmatrix} (d^i)$$
$$= \begin{pmatrix} f^i \\ -d^i \end{pmatrix} + \begin{pmatrix} 0 \\ d^i \end{pmatrix} = \begin{pmatrix} f^i \\ 0 \end{pmatrix}$$

completes the proof of the lemma. □

Proposition 5.47 — *There is a canonical morphism $\mathrm{C}^\bullet(\alpha_f) \to A^\bullet[+1]$ given in degree i by*

$$(0, \beta_f^i) : A'^{i+1} \oplus \mathrm{C}^i(f) \to A^i[+1].$$

Moreover, this morphism is a homotopy equivalence.

Proof. We first explicit the complex $\mathrm{C}^\bullet(\alpha_f)$. We have

$$\mathrm{C}^i(\alpha_f) = A'^i \oplus A^{i+1} \oplus A'^{i+1}$$

The differential is given by

$$\begin{pmatrix} d'^i & f^{i+1} & \mathrm{id}_{A'^{i+1}} \\ 0 & -d^{i+1} & 0 \\ 0 & 0 & -d'^{i+1} \end{pmatrix} : A'^i \oplus A^{i+1} \oplus A'^{i+1} \to A'^{i+1} \oplus A^{i+2} \oplus A'^{i+2}.$$

We define morphisms $u : \mathrm{C}^\bullet(\alpha_f) \to A^\bullet[+1]$ and $v : A^\bullet[+1] \to \mathrm{C}^\bullet(\alpha_f)$ by

$$u^i = (0, \mathrm{id}_{A^{i+1}}, 0) : A'^i \oplus A^{i+1} \oplus A'^{i+1} \to A^{i+1}$$

$$\text{and} \quad v^i = \begin{pmatrix} 0 \\ \mathrm{id}_{A^{i+1}} \\ -f^{i+1} \end{pmatrix} : A^{i+1} \to A'^i \oplus A^{i+1} \oplus A'^{i+1}.$$

The identities

$$(0, \mathrm{id}_{A^{i+2}}, 0) \begin{pmatrix} d'^i & f^{i+1} & \mathrm{id}_{A'^{i+1}} \\ 0 & -d^{i+1} & 0 \\ 0 & 0 & -d'^{i+1} \end{pmatrix} = (0, -d^{i+1}, 0) \quad \text{and}$$

$$\begin{pmatrix} d'^i & f^{i+1} & \mathrm{id}_{A'^{i+1}} \\ 0 & -d^{i+1} & 0 \\ 0 & 0 & -d'^{i+1} \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_{A^{i+1}} \\ -f^{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ -d^{i+1} \\ d'^{i+1} f^{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{A^{i+2}} \\ -f^{i+2} \end{pmatrix} (-d^{i+1}),$$

shows that u and v are indeed morphisms of complexes. Obviously, $u \circ v = \mathrm{id}_{A^\bullet[+1]}$. It remains to show that $u \circ v$ is homotopic to the identity. In degree i, $u \circ v$ is given by the matrix

$$\begin{pmatrix} 0 \\ \mathrm{id}_{A^{i+1}} \\ -f^{i+1} \end{pmatrix} (0, \mathrm{id}_{A^{i+1}}, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id}_{A^{i+1}} & 0 \\ 0 & -f^{i+1} & 0 \end{pmatrix}.$$

We consider the homotopy h given by

$$h^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{A'^i} & 0 & 0 \end{pmatrix} : A'^i \oplus A^{i+1} \oplus A'^{i+1} \to A'^{i-1} \oplus A^i \oplus A'^i.$$

We do the computation:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{A'^{i+1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} d'^i & f^{i+1} & \mathrm{id}_{A'^{i+1}} \\ 0 & -d^{i+1} & 0 \\ 0 & 0 & -d'^{i+1} \end{pmatrix} + \begin{pmatrix} d'^{i-1} & f^i & \mathrm{id}_{A'^i} \\ 0 & -d^i & 0 \\ 0 & 0 & -d'^i \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{A'^i} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d'^i & f^{i+1} & \mathrm{id}_{A'^{i+1}} \end{pmatrix} + \begin{pmatrix} \mathrm{id}_{A'^i} & 0 & 0 \\ 0 & 0 & 0 \\ -d'^i & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{id}_{A'^i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f^{i+1} & \mathrm{id}_{A'^{i+1}} \end{pmatrix} = \mathrm{id}_{A'^i \oplus A^{i+1} \oplus A'^{i+1}} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id}_{A^{i+1}} & 0 \\ 0 & -f^{i+1} & 0 \end{pmatrix}.$$

This finishes the proof of the proposition. □

Corollary 5.48 — *Let $A^\bullet$ be a complex. Then $\mathrm{C}^\bullet(\mathrm{id}_A)$ is contractible, i.e., its identity morphism is homotopic to the zero morphism. In particular, $\mathrm{H}^i(\mathrm{C}^\bullet(\mathrm{id}_A)) = 0$.*

Proof. Apply the previous proposition for $f : 0 \to A^\bullet$ and remark that $\alpha_f = \mathrm{id}_{A^\bullet}$. ☐

Proposition 5.49 — *For every $i \in \mathbb{Z}$, we have an exact sequence*

$$\mathrm{H}^i(A) \xrightarrow{\mathrm{H}^i(f)} \mathrm{H}^i(A') \xrightarrow{\mathrm{H}^i(\alpha_f)} \mathrm{H}^i(\mathrm{C}(f))$$

Proof. That the composition is zero follows from Lemma 5.46. We need to show that $\mathrm{im}(\mathrm{H}^i(f)) \simeq \ker(\mathrm{H}^i(\alpha_f))$.

The image of $\mathrm{H}^i(f)$ can be written as

$$\frac{\mathrm{im}[f^i : \ker(d^i) \to \ker(d'^i)]}{\mathrm{im}[f^i : \ker(d^i) \to \ker(d'^i)] + \mathrm{im}(d'^{i-1})}$$

On the other hand, the kernel of $\ker(d'^i) \to \mathrm{H}^i(\mathrm{C}(f))$ is given by the fiber product

$$\ker(d'^i) \times_{\ker(d_f^i)} \mathrm{im}(d_f^{i-1}) \simeq \ker(d'^i) \times_{\mathrm{C}^i(f)} \mathrm{im}(d_f^{i-1}).$$

In other words, it is the intersection in $A'^i \oplus A^{i+1}$: $(\ker(d'^i) \oplus 0) \cap \mathrm{im}(d_f^{i-1})$. This contains $\mathrm{d}_f^{i-1}(A'^{i-1} \oplus 0)$ and $\mathrm{d}_f^{i-1}(0 \oplus \ker(d^i))$. Hence, the fiber product is given by $\mathrm{im}(d'^{i-1}) + f^i(\ker(d^i))$. This proves the proposition. ☐

Corollary 5.50 — *Let $f : A^\bullet \to A'^\bullet$ be a morphism of complexes. Then, we have a long exact sequence*

$$\dots \to \mathrm{H}^i(A) \to \mathrm{H}^i(A') \to \mathrm{H}^i(\mathrm{C}(f)) \to \mathrm{H}^{i+1}(A) \to \mathrm{H}^{i+1}(A') \to \dots$$

Lemma 5.51 — *Let*

$$0 \to A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \to 0$$

be an exact sequence of complexes in an abelian category $\mathcal{A}$. Then there is a natural morphism of complexes $\mathrm{C}^\bullet(a) \to C^\bullet$ given in degree i by

$$(b^i, 0) : B^i \oplus A^{i+1} \to C^i.$$

Moreover, this morphism is a quasi-isomorphism.

Proof. We call d_A, d_B and d_C the differentials in A, B and C. The existence of a morphism follows from the computation

$$d_C^i \circ (b^i, 0) = (d_C^i \circ b^i, 0) = (b^{i+1}) \circ d_B^i, 0) = (b^{i+1}, 0) \begin{pmatrix} d_B^i & a^{i+1} \\ 0 & -d_A^{i+1} \end{pmatrix}$$

which holds as $b^{i+1} \circ a^{i+1} = 0$.

Now, we need to show that

$$\frac{\ker(d_a^i : B^i \oplus A^{i+1} \to B^{i+1} \oplus A^{i+2})}{\mathrm{im}(d_a^{i-1} : B^{i-1} \oplus A^i \to B^i \oplus A^{i+1})} \simeq \frac{\ker(d_C^i : C^i \to C^{i+1})}{\mathrm{im}(d_C^{i-1} C^{i-1} \to C^i)}$$

We remark that both

$$\ker(d_a^i : B^i \oplus A^{i+1} \to B^{i+1} \oplus A^{i+2}) \quad \text{and} \quad \operatorname{im}(d_a^{i-1} : B^{i-1} \oplus A^i \to B^i \oplus A^{i+1})$$

contain $\operatorname{im}(\begin{pmatrix} a^i \\ -d_A^i \end{pmatrix} : A^i \to B^i \oplus A^{i+1})$. Thus, we are reduced to show that We have an exact sequence

$$0 \to A^{i+1} \to \operatorname{coker}(A^i \to B^i \oplus A^{i+1}) \to C^i \to 0$$

It follows that the obvious map

$$\frac{\ker(d_a^i : B^i \oplus A^{i+1} \to B^{i+1} \oplus A^{i+2})}{A^i} \to C^i$$

is an isomorphism to $\ker(d_C^i)$.

Similarly, we show that

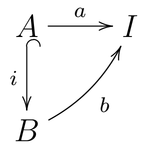
$$\frac{\operatorname{im}(d_a^{i-1} : B^{i-1} \oplus A^i \to B^i \oplus A^{i+1})}{A^i} \to C^i$$

gives an isomorphism to $\operatorname{im}(d_C^{i-1})$. □

5.8. **Injective and projective objects in an abelian category.**

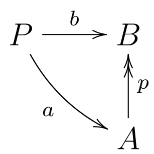
We fix an abelian category $\mathcal{A}$.

Definition 5.52 — *An object $I \in \operatorname{ob}(\mathcal{A})$ is* injective *if it satisfies the following property. For any injective morphism $i : A \hookrightarrow B$ in $\mathcal{A}$ and any $a : A \to I$, there exists a morphism $b : B \to I$ making the triangle*



commutative.

An object P in $\mathcal{A}$ is projective *if and only if it satisfies to the following property. For any surjective morphism $p : A \twoheadrightarrow B$ in $\mathcal{A}$ and any $b : P \to B$, there exists a morphism $a : P \to A$ making the triangle*



commutative.

Remark 5.53 — An object of $\mathcal{A}$ is injective (resp. projective) if and only if it is projective (resp. injective) as an object of the opposite category $\mathcal{A}^\circ$. In the sequel, we only consider injective objects. Any statement concerning injective objects has a dual statement concerning projective objects.

Definition 5.54 — *We say that $\mathcal{A}$ has enough injectives (resp. projectives) if for every object A in $\mathcal{A}$, there is an injection $A \hookrightarrow I$ (resp. a surjection $P \twoheadrightarrow A$) with I injective (resp. P projective).*

Example 5.55 — Let k be a field and $\mathcal{M}od(k)$ be the abelian category of k-vector spaces. Then every k-vector space V is injective and projective. In particular, $\mathcal{M}od(k)$ has enough injectives and projectives.

Let A is a commutative ring. Then every free A-module M is a projective object of $\mathcal{M}od(A)$. It follows that $\mathcal{M}od(A)$ has enough projectives. We will see later that $\mathcal{M}od(A)$ has also enough injectives.

Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is said to be exact if it takes exact sequences to exact sequences. We say that F is left exact if for any exact sequence $0 \to A \to B \to C$, $0 \to F(A) \to F(B) \to F(C)$ is exact. Dually, we say that F is right exact if for any exact sequence $A \to B \to C \to 0$, $F(A) \to F(B) \to F(C) \to 0$ is exact. Then a functor F is exact if and only if it is left and right exact.

Lemma 5.56 — *Let $\mathcal{A}$ be an abelian category and I an object of $\mathcal{A}$. Then the following is equivalent:*

(i) *I is an injective object,*
(ii) *The functor* $\hom_{\mathcal{A}}(-, I) : \mathcal{A} \to \mathrm{Ab}$ *is right exact,*
(iii) *The functor* $\hom_{\mathcal{A}}(-, I) : \mathcal{A} \to \mathrm{Ab}$ *is exact.*

Proof. We first show that $\hom(-, I)$ is left exact regardless of I being injective. Let

$$A \to B \to C \to 0$$

be an exact sequence. We want to show that

$$0 \to \hom(C, I) \to \hom(B, I) \to \hom(A, I)$$

is exact. That $\hom(C, I) \to \hom(B, I)$ is injective follows from the fact that $B \to C$ is surjective. To show exactness in the middle, let $b : B \to I$ be such that $b \circ (A \to B)$ is zero. Then, b factors trough $\mathrm{coker}(A \to B)$ which canonically isomorphic to C. Hence, b is in the image of $\hom(C, I) \to \hom(B, I)$.

It follows that $\hom_A(-, I)$ is exact if and only if it is right exact. It remains to show that (i)⇔(iii). Let $0 \to A \to B \to C \to 0$ be a short exact sequence. To show that

$$0 \to \hom(C, I) \to \hom(B, I) \to \hom(A, I) \to 0$$

it remains to show that $\hom(B, I) \to \hom(A, I)$ is surjective. But this is exactly the definition of an injective object. □

5.9. Injective and projective resolutions.

Let $\mathcal{A}$ be an abelian category.

Definition 5.57 — *Let A be an object of $\mathcal{A}$.*

a) *A right resolution of A is a complex $R^\bullet$ in $\mathcal{A}$ concentrated in positive degrees (i.e., $R^i = 0$ for $i < 0$) together with a morphism (called, an augmentation) $A \to R^0$ such that:*

(1) *the composition $A \to R^0 \to R^1$ is zero,*
(2) *the natural morphism $A \to \mathrm{H}^0(R)$ induced by the augmentation is an isomorphism,*
(3) *$\mathrm{H}^i(R) = 0$ for $i \neq 0$.*

In other words,

$$0 \to A \to R^0 \to R^1 \to \cdots \to R^i \to \ldots$$

is a long exact sequence.

b) *Dually, we have the notion of left resolution of A. It is a complex $L^\bullet$ concentrated in negative degrees (i.e., $L^i = 0$ for $i > 0$) together with a morphism (called, a coaugmentation) $L^0 \to A$ such that the dual of the properties above are satisfied.*

Definition 5.58 — *Let A be an object of $\mathcal{A}$. An* injective resolution *of A is a right resolution $I^\bullet$ such that I^i is injective for all $i \in \mathbb{Z}$. Dually, a* projective resolution *of A is a left resolution $P^\bullet$ such that P^i are projective for all $i \in \mathbb{Z}$.*

Lemma 5.59 — *Assume that $\mathcal{A}$ has enough injectives. Then every object of $\mathcal{A}$ has an injective resolution.*

Proof. Let $A \in \mathcal{A}$. We may find an injective morphism $A \hookrightarrow I^0$ with I^0 injective. Also we may find $I^0/A \hookrightarrow I^1$ with I^1 injective. We let $d^0 : I^0 \to I^1$ be the obvious morphism. By induction, we may assume that I^i is defined and $d^{i-1} : I^{i-1} \to I^i$. We the chose an injection $\mathrm{coker}(d^{i-1}) \hookrightarrow I^{i+1}$ with I^{i+1} injective and take for $d^i : I^i \to I^{i+1}$ the obvious morphism. It is then clear that the complex $I^\bullet$ is an injective resolution of A. □

The next statement shows that injective resolutions are functorial up to homotopy.

Proposition 5.60 — *Let $a : A \to B$ be a morphism in $\mathcal{A}$. Let $R^\bullet$ be a right resolution of A and $J^\bullet$ an injective resolution of B. Then, there exist a morphism of complexes $f : R^\bullet \to J^\bullet$ compatible with the augmentations, i.e., such that*

$$\begin{array}{ccc} A & \longrightarrow & R^0 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f^0} \\ B & \longrightarrow & J^0 \end{array}$$

commutes. Moreover, f is unique up to homotopy.

Proof. We construct $f^i : R^i \to J^i$ by induction. We find f^0 using that $A \to R^0$ is injective. Assume that we have constructed

$$\begin{array}{ccc} R^{i-1} & \xrightarrow{d_R^{i-1}} & R^i \\ {\scriptstyle f^{i-1}}\downarrow & & \downarrow{\scriptstyle f^i} \\ J^{i-1} & \xrightarrow{d_J^{i-1}} & J^i \end{array}$$

Then we need to construct f^{i+1} such that

$$\begin{array}{ccc} \mathrm{coker}(d_R^{i-1}) & \longrightarrow & R^{i+1} \\ \downarrow & & \vdots\, {\scriptstyle f^{i+1}} \\ \mathrm{coker}(d_J^{i-1}) & \longrightarrow & J^{i+1}. \end{array}$$

This is again possible as J^{i+1} is an injective object and $\mathrm{coker}(d_R^{i-1}) \to R^{i+1}$ is an injective morphism.

Next we show uniqueness up to homotopy. Let $f,\ g : R^\bullet \to J^\bullet$ be two morphisms compatible with the augmentations. We will construct inductively a homotopy $(h^i :$

$R^i \to J^{i-1})_{i\in\mathbb{Z}}$ between f and g. Obviously, we have take $h^i = 0$ for $i \leq 0$. We define h^1 as follows. The difference $f^0 - g^0$ vanishes on A. Hence, it induces a morphism $R^0/A \to J^0$. As $R^0/A \to R^1$ is injective, this morphism can be extended to $h^1 : R^1 \to J^0$.

Assume that h^i was constructed so that $f^{i-1} - g^{i-1} = h^i d_R^{i-1} + d_J^{i-2} h^{i-1}$. To construct h^i, we consider $f^i - g^i - d_J^{i-1} h^i : R^i \to J^i$. As

$$\begin{aligned}(f^i - g^i - d_J^{i-1}h^i)d_R^{i-1} &= d_J^{i-1}(f^{i-1} - g^{i-1}) - d_J^{i-1}h^i d_R^{i-1} \\ &= d_J^{i-1}(h^i d_R^{i-1} + d_J^{i-2}h^{i-1}) - d_J^{i-1}h^i d_R^{i-1} = 0\end{aligned}$$

we get a morphism $\operatorname{coker}(d_R^{i-1}R^{i-1} \to R^i) \to J^i$. As J^i is an injective object, and because $\operatorname{coker}(d_R^{i-1}R^{i-1} \to R^i) \to R^{i+1}$ is an injective morphism, we can extend the latter to $h^{i+1} : R^{i+1} \to J^i$. It is easily checked that this completes the inductive step. □

Corollary 5.61 — *Let A be an object of $\mathcal{A}$. Let $I^\bullet$ and $I'^\bullet$ be two injective resolutions of A. Then there exists a morphism of complexes $I^\bullet \to I'^\bullet$ compatible with the augmentation and which is a homotopy equivalence.*

5.10. **Derived functors.**

Let $\mathcal{A}$ be an abelian category having enough injectives and

For each object $A \in \mathcal{A}$, choose an injective resolution $I^\bullet(A)$ (this is possible as $\mathcal{A}$ has enough injectives). For $a : A \to B$ a morphism in $\mathcal{A}$, choose a morphism of complexes $i(a) : I^\bullet(A) \to I^\bullet(B)$ compatible with the augmentation morphism. Such a morphism exists and is unique up to homotopy. Moreover, given $a : A \to B$ and $b : B \to C$, the morphisms of complexes $i(b) \circ i(a)$ and $i(b \circ a)$ are homotopic.

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. For $A \in \mathcal{A}$, we set $\mathrm{R}^i F(A) := \mathrm{H}^i(F(I(A))$. For $a : A \to B$, we take for $\mathrm{R}^i F(a) : \mathrm{R}^i F(A) \to \mathrm{R}^i F(B)$ the morphism $\mathrm{H}^i(F(i(a)))$. If $b : B \to C$ is another morphism, we have $\mathrm{R}^i F(b \circ a) = \mathrm{R}^i F(b) \circ \mathrm{R}^i F(a)$ as $F(i(b \circ a))$ and $F(i(b)) \circ F(i(a))$ are homotopic. Thus we have defined a family of functors $\mathrm{R}^i F : \mathcal{A} \to \mathcal{B}$.

The dual construction yields another family of functors $\mathrm{L}^j F : \mathcal{A} \to \mathcal{B}$.

Definition 5.62 — *The functor* $\mathrm{R}^i F$ *is called the i-th* right derived functor *associated to F. The functor* $\mathrm{L}^j F$ *is called the j-th* left derived functor*.*

It is easy to see that right and left derived functors are well defined up to a unique isomorphisms of functors. Obviously, $\mathrm{R}^i F(A) = 0$ for $i < 0$ and $\mathrm{L}^j F(A) = 0$ for $j < 0$. Also, $\mathrm{R}^i F$ and $\mathrm{L}^j F$ are additive.

Proposition 5.63 — *Assume that $F : \mathcal{A} \to \mathcal{B}$ is left exact. Then $F \simeq \mathrm{R}^0 F$. Moreover, given a short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, there is a natural long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to \mathrm{R}^1 F(A) \to \mathrm{R}^1 F(B) \to \mathrm{R}^1 F(C) \to \mathrm{R}^2 F(A) \to \dots$$

When F is exact, the $\mathrm{R}^i F(A) = 0$ for all A and $i > 0$.

Proof. As $0 \to A \to I^0(A) \to I^1(A)$ is exact, and F is left exact, we deduce that $0 \to F(A) \to F(I^0(A)) \to F(I^1(A))$ is also exact. It follows that $F(A) \simeq \mathrm{R}^0F(A)$.

Now, given an injective resolution $I^\bullet$ of A, we may construct an injective resolution $J^\bullet$ of B together with a morphism $I^\bullet \to J^\bullet$ which is injective degree-wise. We then define $K^\bullet = J^\bullet/I^\bullet$. We claim that $K^\bullet$ is an injective resolution of C.

That K^i is injective is clear as the monomorphism $I^i \to J^i$ necessarily split. Moreover, by ??, we have a long exact sequence

$$\dots \to \mathrm{H}^i(I) \to \mathrm{H}^i(J) \to \mathrm{H}^i(K) \to \mathrm{H}^{i+1}(I) \to \dots$$

This shows that $\mathrm{H}^0(K) = C$ and $\mathrm{H}^i(K) = 0$ for $i \neq 0$.

Now applying F, we get degree-wise split exact sequence of complexes

$$0 \to F(I) \to F(J) \to F(K) \to 0$$

Using again ??, we get our exact sequence. □

Definition 5.64 — *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. An object $E \in \mathcal{A}$ is called F-*acyclic *if* $\mathrm{R}^iF(E) = 0$ *for* $i > 0$.

Let A be an object of $\mathcal{A}$. An F-acyclic resolution of A is a right resolution $E^\bullet$ of A where E_i is F-acyclic for every $i \in \mathbb{Z}$.

Proposition 5.65 — *Let A be an object of $\mathcal{A}$ and $E^\bullet$ an F-acyclic resolution of A. There are canonical isomorphisms* $\mathrm{R}^iF(A) \simeq \mathrm{H}^i(F(E))$.

Proof. We may find an injective resolution $I^\bullet$ of A with a morphism of left resolutions $E^\bullet \to I^\bullet$ which is injective degree-wise. Let $K^\bullet = I^\bullet/E^\bullet$. We claim that

$$0 \to F(E^\bullet) \to F(I^\bullet) \to F(K^\bullet) \to 0$$

is degree-wise a short exact sequence of complexes. This follows immediately from the fact that $\mathrm{R}^1F(E^i) = 0$ for all i.

Thus, it suffices to show that $F(K^\bullet)$ has zero cohomology. Remark that K^i is F-acyclic for every i. We first show by induction on i that $\ker(d^i_K)$ is F-acyclic for all i. This is true for i small enough.

Assume the induction hypothesis for some $i \in \mathbb{Z}$. We then use the exact sequence

$$0 \to \ker(d^i_K) \to K^i \to \ker(d^{i+1}_K) \to 0.$$

to get that $\ker(d^{i+1}_K)$ is also F-acyclic.

Now, we check that $\mathrm{H}^i(F(K)) = 0$. The image of $\mathrm{d}^{i-1}_{F(K)} : F(K^{i-1}) \to F(K^i)$ is identified with the cokernel of $\ker(F(\mathrm{d}^{i-1}_K)) \to F(K^{i-1})$ which is by left exactness of F, isomorphic to the cokernel of $F(\ker(d^{i-1}_K)) \to F(K^{i-1})$. As $\ker(d^{i-1}_K)$ is F-acyclic, this cokernel is simply $F(\mathrm{coker}(\ker(d^{i-1}_K) \to K^{i-1}))$ which can be identified with $F(\mathrm{im}(d^{i-1}_K))$. In other words, the canonical morphism

$$F(\mathrm{im}(d^{i-1}_K)) \simeq \mathrm{im}(d^{i-1}_{F(K)}).$$

On the other hand, using that F is left exact, we have $\ker(d^i_{F(K)}) \simeq F(\ker(d^i_K))$. We are done as $\ker(d^i_K) = \mathrm{im}(d^{i-1}_K)$. □